Treasure game*

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Abstract

We study the investment choice of two agents in a R&D race where the necessary threshold investment for success is uncertain. The race is modeled as a multistage game with observed previous actions where the player's probability of success depends only on his investment in that period. We provide a complete characterization of a symmetric equilibrium of this game. We find that the investment process can be divided in two stages: in the first stage cooperative behavior of players can be supported as an equilibrium; in the second stage there is overinvesting in comparison with cooperative equilibrium.

Keywords: R&D, uncertainty, search.

JEL classifications: O32.

1 Introduction

In this paper we analyze a dynamic model where two players search for a treasure hidden somewhere on an island. Each player has to decide how much to dig each period. If the treasure is found, the game ends immediately. If the treasure is not found, the size of the island shrinks to the initial size minus the part that has been dug. Next period players have to make their digging decisions again. We assume that

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digging is costly and players can observe what the other player dug in the previous periods.

There are many examples of this situation: two detectives (two police units) are looking for a criminal; two journalists are looking for a movie star in the city hotels; R&D race between two companies or two laboratories where digging is an intensity of research.

We assume that if both players find the treasure simultaneously (dig the same part of the island), each of them incurs costs but the treasure will be destroyed (players do not get any treasure). This assumption is standard in the R&D literature (see Chatterjee and Evans, 2004). It can be justified on the ground that if both players discover the treasure simultaneously, they will be involved in the fierce competition afterwards and run out of any profit. A good example of this situation is 1960s Lockheed and Douglas jet development competition.[1] Many examples of simultaneous discoveries in science can be found in Merton (1973).

PUBLIC GOOD EXAMPLES!

We analyze a symmetric situation and find the equilibrium "digging plans". First, there is no duplication.

Second, the investment process can be divided in two stages: in the first stage cooperative investment behavior of players can be supported as an equilibrium; in the second stage there is overinvestment in comparison with cooperative equilibrium.

Our paper is related to the individual search literature; see Ross (1983) and Gittins (1989). However, players are assumed dig strategically in our model.

Fershtman and Rubinstein (1997) consider an interactive model in which two players search for a single hidden treasure in one of a given set of labeled boxes. Each player simultaneously choose how many search units he wants to employ. Search strategies are selected by the players after observing each other choice regarding the number of search units. Employing each unit is costly, while the actual searching is not. Contrary, in our model each player can employ only one unit and the costs are

[1]For more detail see The Economist, 1985; and Chatterjee and Evans, 2004.

due to the actual searching.

Chatterjee and Evans (2004) analyze R&D model which is similar to ours. They allow each of two firms to observe the other's past choices and search strategically. Their firms have to choose between two research projects. We have only one research project in their notation. Their model in a sense is complementary to our model. While agents in their model decide which area to search in (the size is predetermined), agents in our model decide how much area to search in (the location has no importance).

The paper is organized as follows. The model is developed and the main Bellman equation is described in section 2. We find the solution of the Bellman equation and discuss properties of this solution in Section 3. Section 4 concludes.

2 The Model

Consider the following situation. There are two risk-neutral players, Linda (L) and Jerry (J). They are looking for treasure. The person who discovers the treasure gets payoff of one and the other person receives payoff of p, where $0 \leq p \leq 1$. Two extreme cases $p = 0$ and $p = 1$ correspond to private and public good situations. If $0 < p < 1$, then we have a good which has public and private value.

The treasure is hidden somewhere on the island. The size of the island is x_0 and the treasure has equal chances to be at any part of the island.[2] Players have to dig the island in order to find the treasure. Digging is costly. We assume that the cost function is linear in the amount of digging.

The time is discrete and both players have a discount factor, δ. In every period, both players have to decide simultaneously how much to dig in the current period. We assume that a player can see what the other player have dug so far before making digging plans for the next period. If the treasure is not found in the current period,

[2]We focus our attention on uniform distribution because this represents a situation when there is no information about which parts of the island are more or less likely than others. Note that this is the worst situation from the players point of view.

the island shrinks. A new size of the island will be equal to the previous island size minus the dug part.

If both players decide jointly to dig more than the current size of the island then both incur the costs but do not get any treasure. It means that in the equilibrium players will never duplicate.

If $0 \leq I_1^i \leq x$ is chosen to dig by player i in period 1, then the treasure is either found in the first period (with probability $(I_1^L + I_1^J)/x$) or players will have to dig a smaller part of the island, $z_1 = x - I_1^L - I_1^J$, with probability $1 - (I_1^L + I_1^J)/x$, in the next period. If the treasure is found by Linda then she gets 1, while if the treasure is found by Jeremy Linda gets p.

Linda has to solve the following Bellman equation:

$$V^L(x) = \max_{0 \leq I_1^L \leq x - I_1^J} \{-I_1^L + I_1^L/x + pI_1^J/x + \delta(1 - (I_1^L + I_1^J)/x)V^L(x - I_1^L - I_1^J)\}, \quad (1)$$

where the first term in this equation describes Linda's costs of investment. The second term is Linda's expected value from finding the treasure in the first period, the third term is Linda's expected value from Jeremy's finding the treasure in the first period. The last term is Linda's expected value from winning after the first period.

3 Analysis of the Model

Define the part of the island which Linda does not dig in the first period by

$$y^L := x - I_1^L$$

and the part of the island which both players do not dig in the first period (the remaining part of the island) by

$$z := x - I_1^L - I_1^J.$$

Then equation (1) can be rewritten in the following way

$$\begin{gathered} BV^L(z) := V^L(x) = \\ \max_{I_1^J \leq y^L \leq x} \{-(x - y^L) + (x - y^L)/x + pI_1^J/x + \delta z V^L(z)/x\}. \end{gathered} \quad (2)$$

We are interested to derive Linda's value of the game, $V^L(x)$. From economic context it is clear that

Assumption 1.

$$0 \le V^L(x) \le 1 \ \ \textit{for any } x; \tag{3}$$

and

$$0 \le z \le x. \tag{4}$$

It is convenient to introduce function

$$\Psi^L(x) := xV^L(x). \tag{5}$$

Of course, from (3) and (5)

$$\Psi^L(x) \ge 0 \ \text{ for any x.} \tag{6}$$

Then in symmetric equilibrium, $I_1^L = I_1^J = I$, equation (2) becomes

$$\Psi^L(x) = \max_{I \le y^L \le x} \{(1-x)(x-y^L) + pI + \delta \Psi^L(z)\} =: B\Psi^L(z). \tag{7}$$

Lemma 1. *If $\delta < 1$, the operator on the right hand side of equation (7) is a contraction operator. Therefore, equation (7) has a unique solution, Ψ^L, that can be obtained as the limit of the following sequence $\{\Psi_k^L\}$, where*

$$\Psi_0^L \equiv 0, \qquad \Psi_k^L := B\Psi_{k-1}^L \quad k = 1, 2, ... \quad . \quad \square \tag{8}$$

Proof. See the Appendix.

3.1 Construction of the sequence $\{\Psi_k^L\}$ and $\{V_k^L\}$

Note that based on Lemma 1 we can construct the sequence $\{\Psi_k^L\}$. It is equivalent to using the backward induction argument. In order to simplify the exposition, all players indexes, L and J, will be dropped.

3.1.1 Construction of Ψ_1 and V_1.

Let us start from the end of the investment process. What will be the value of the game, if players can dig the whole island in at most one period? Since duplication is not feasible players will invest together $I^L + I^J = x$ in the equilibrium. In the symmetric equilibrium

$$y_1 = x/2, \tag{9}$$

or

$$y = x. \tag{10}$$

Suppose that (9) holds in the equilibrium, then

$$\Psi_1(x) = \max_{I \le y \le x} \{(1-x)(x-y) + pI\}. \tag{11}$$

Therefore from assumption 2 and condition (9), equation (11) becomes

$$\Psi_1(x) = \begin{cases} x(1+p-x)/2, & \text{if} \quad x < u_1 = 1+p, \\ 0, & \text{if} \quad x \ge u_1 = 1+p, \end{cases} \tag{12}$$

where u_1 is a positive root of the polynomial $x(1+p-x)/2$. Note that

$$x(1+p-x)/2 = -\frac{1}{2}(1+p-x)^2 + \frac{1+p}{2}(1+p-x). \tag{13}$$

If the players can dig the island in at most one period, then the symmetric equilibrium will look like

$$y(x) = \begin{cases} x/2, & \text{if} \quad x < u_1, \\ 0, & \text{if} \quad x \ge u_1. \end{cases} \tag{14}$$

The optimal first-period digging is independent from the discount factor, δ, because there is no delay here.

Define a sequence $\{V_k\}$, where $V_k(x) := \Psi_k(x)/x$, for any $x > 0$. V can be obtained as the limit of the sequence $\{V_k\}$. Then

$$V_1(x) = \begin{cases} (1+p-x)/2, & \text{if} \quad x < u_1, \\ 0, & \text{if} \quad x \geq u_1. \end{cases} \tag{15}$$

3.1.2 Construction of Ψ_2 and V_2.

What will be the value of the game, if players can dig the whole island in at most two periods? It is obvious that there are three possibilities for different island sizes. In the first case, the players will dig the whole island in just one period. In this case, $V_2 = V_1$. In the second case, the players will dig the island in exactly two periods. In symmetric equilibrium, $y^L = y^J = y$ and therefore

$$z = 2y - x > 0. \tag{16}$$

In this case, $V_2 > V_1$. Two-period digging gives higher payoffs than one-period digging. Finally, the players can find digging to be too costly and don't dig at all. In this case, $V_2 = V_1 = 0$.

Let us start from the first case. Then,

$$V_1(x) \geq V_2(x), \tag{17}$$

or

$$\Psi_1(x) \geq \Psi_2(x), \tag{18}$$

for small x. The intuition for this observation is as follows. Players want to dig the whole island, if it is very small, and they don't want to wait for the second period. In other words, the inequality (18) must hold on the left from the indifference point $\Psi_1(t_1) = \Psi_2(t_1)$. Let us find this indifference point t_1.

First, we have to construct $\Psi_2(x)$. From (7),

$$\Psi_2(x) = \max_{I \leq y \leq x} \{(1-x)(x-y) + pI + \delta\Psi_1(z)\}. \tag{19}$$

The necessary condition for the optimal value y to be an inferior point in $[0, x]$ is

$$-(1-x) + \delta\Psi_1'(z) = 0. \tag{20}$$

From the condition (20) and the expression (12), it follows

$$-(1-x)+\delta\left(\frac{1+p}{2}-z\right)=0.$$

Consequently,

$$z(x)=\frac{2(x-1)+(1+p)\delta}{2\delta}. \tag{21}$$

Therefore from (16),

$$y=\frac{x+z(x)}{2}=\frac{2(1+\delta)x-2+(1+p)\delta}{4\delta}. \tag{22}$$

Plugging (22) into equation (19), we obtain

$$\Psi_2(x)=\begin{cases}\Psi_1(x), & \text{if} \quad x<t_1,\\ -\frac{1}{2}(1+p-x)^2+\frac{\delta(1+p)+2p}{4\delta}(1+p-x)+\frac{\delta^2(1+p)^2-4p^2}{8\delta}, & \text{if} \quad t_1\le x<u_2,\\ 0, & \text{if} \quad x\ge u_2,\end{cases} \tag{23}$$

where u_2 is a positive root of the polynomial

$$-\frac{1}{2}(1+p-x)^2+\frac{\delta(1+p)+2p}{4\delta}(1+p-x)+\frac{\delta^2(1+p)^2-4p^2}{8\delta},$$

and t_1 is the island size such that the players are indifferent between digging the island in two periods, or in one period:

$$V_1(t_1)=V_2(t_1), \tag{24}$$

or

$$\Psi_1(t_1)=\Psi_2(t_1). \tag{25}$$

From (13) and (23), it is equivalent to

$$\frac{\delta(1+p)-2p}{4\delta}(1+p-t_1)=\frac{\delta^2(1+p)^2-4p^2}{8\delta}. \tag{26}$$

Note that if the following condition holds

$$\delta=\frac{2p}{1+p}, \tag{27}$$

then equation (26) becomes equivalence for any t_1. Therefore, condition (27) describes the region on δ and p plane where players dig the whole island in one period.

The following expression is the solution of the equation (26)

$$t_1 = 1 - \frac{\delta(1+p)}{2}. \tag{28}$$

Consider the second case. The players will dig the island in two periods instead of one if

$$V_1(x) < V_2(x) \tag{29}$$

and

$$V_2(x) \geq 0. \tag{30}$$

It is equivalent to

$$\Psi_1(x) < \Psi_2(x) \tag{31}$$

and

$$\Psi_2(x) \geq 0. \tag{32}$$

From (13) and (23), we obtain

$$\frac{\delta(1+p) - 2p}{4\delta}(1+p-x) < \frac{\delta^2(1+p)^2 - 4p^2}{8\delta}. \tag{33}$$

Note that it must be

$$x > t_1 = 1 - \frac{\delta(1+p)}{2}. \tag{34}$$

Hence

$$\delta(1+p) - 2p > 0, \tag{35}$$

if the players dig the whole island in two periods.

Therefore, if the players can dig the island in at most two periods, the symmetric equilibrium will look like

$$y(x) = \begin{cases} x/2, & \text{if} \quad x < t_1, \\ \frac{2(1+\delta)x - 2 + (1+p)\delta}{4\delta}, & \text{if} \quad t_1 \leq x < u_2, \\ x, & \text{if} \quad x \geq u_2. \end{cases} \tag{36}$$

Hence,

$$V_2(x) = \begin{cases} V_1, & \text{if} \quad x < t_1, \\ \Psi_2(x)/x, & \text{if} \quad t_1 \le x < u_2, \\ 0, & \text{if} \quad x \ge u_2. \end{cases} \tag{37}$$

3.1.3 Construction of Ψ_k and V_k.

Consider $\Psi_2(x)$ and $\Psi_1(x)$. Note that they are polynomials in the following form:

$$\Psi_k(x) = -\frac{a_k}{2}(1+p-x)^2 + b_k(1+p-x) + c_k, \quad k = 1, 2, \tag{38}$$

where

$$a_1 = 1, \quad b_1 = (1+p)/2, \quad c_1 = 0;$$

and

$$a_2 = 1, \quad b_2 = \frac{\delta(1+p)+2p}{4\delta}, \quad c_2 = \frac{\delta^2(1+p)^2 - 4p^2}{8\delta}.$$

Using (7), Ψ_k can be described in general by the following equation

$$\Psi_k(x) = \max_{I \le y \le x} \{(1-x)(x-y) + pI + \delta\Psi_{k-1}(z)\} =: B\Psi_{k-1}(z), \tag{39}$$

where $\Psi_k(x) \ge 0$ and $\Psi_k(z) \ge 0$.

It is easy to see that if a function Ψ_{k-1} in (39) is polynomial, the next function Ψ_k is also polynomial, because it is equal to $B\Psi_{k-1}$. Note that if Ψ_{k-1} is a quadratic polynomial, then Ψ_k is a quadratic polynomial as well. Let us find a_k, b_k, and c_k for any k.

From condition (20) it follows

$$(1-x) = \delta\Psi'_{k-1}(2y-x). \tag{40}$$

Therefore from (38) and (20),

$$z(x) = \frac{\delta((1+p)a_{k-1} - b_{k-1}) - (1-x)}{\delta a_{k-1}}. \tag{41}$$

Using (16), we obtain

$$y_1 = \frac{x(1+\delta a_{k-1}) + \delta((1+p)a_{k-1} - b_{k-1}) - 1}{2\delta a_{k-1}}. \tag{42}$$

Hence

$$\Psi_k(x) = (1+p-x)(x - \frac{x(1+\delta a_{k-1}) + \delta((1+p)a_{k-1} - b_{k-1}) - 1}{2\delta a_{k-1}}) + \delta \Psi_{k-1}(z_1). \quad (43)$$

Finally, we have

$$a_k = 1, \quad b_k = \frac{b_{k-1} + p/\delta}{2}, \quad c_k = \frac{\delta^2 b_{k-1}^2 - p^2}{2\delta} + \delta c_{k-1}. \quad (44)$$

Proposition 1. *The system of difference equations (44) has the following solution*

$$b_k = \frac{1+p}{2^k} + \frac{p}{\delta}\left(1 - \frac{1}{2^{k-1}}\right), \quad (45)$$

and

$$c_k = p(1+p-2p/\delta)\left(\frac{(2\delta)^{k-1} - 1}{(2\delta - 1)2^{k-1}}\right) + (1+p-2p/\delta)^2 \left(\frac{(4\delta)^{k-1} - 1}{(4\delta - 1)4^{k-1}}\right)\delta/2. \quad (46)$$

Proof. See the Appendix.

Corollary 1. *b_k is a decreasing function of k, if $k \geq 2$.*

Proof. Note that if players want to finish the project in two or more periods, then condition (35) must hold. It follows from formula (45) and condition (35) that b_k is a decreasing function of k. **End of proof.**

Define the island size, t_k, such that the players are indifferent between digging the island in k or in $k+1$ periods, or

$$V_k(t_k) = V_{k+1}(t_k), \quad (47)$$

or

$$\Psi_k(t_k) = \Psi_{k+1}(t_k). \quad (48)$$

Define u_k to be a positive root of the polynomial Ψ_k. Then

Proposition 2. *The players' indifference points are*

$$t_k = 1 + p + \frac{c_{k+1} - c_k}{b_{k+1} - b_k}. \quad (49)$$

Polynomial $\Psi_k(x)$ is positive for any $x \leq u_k$, where

$$u_k = 1 + p - b_k + \sqrt{b_k^2 + 2c_k}. \quad (50)$$

Proof. It follows from equations (48) and (38).

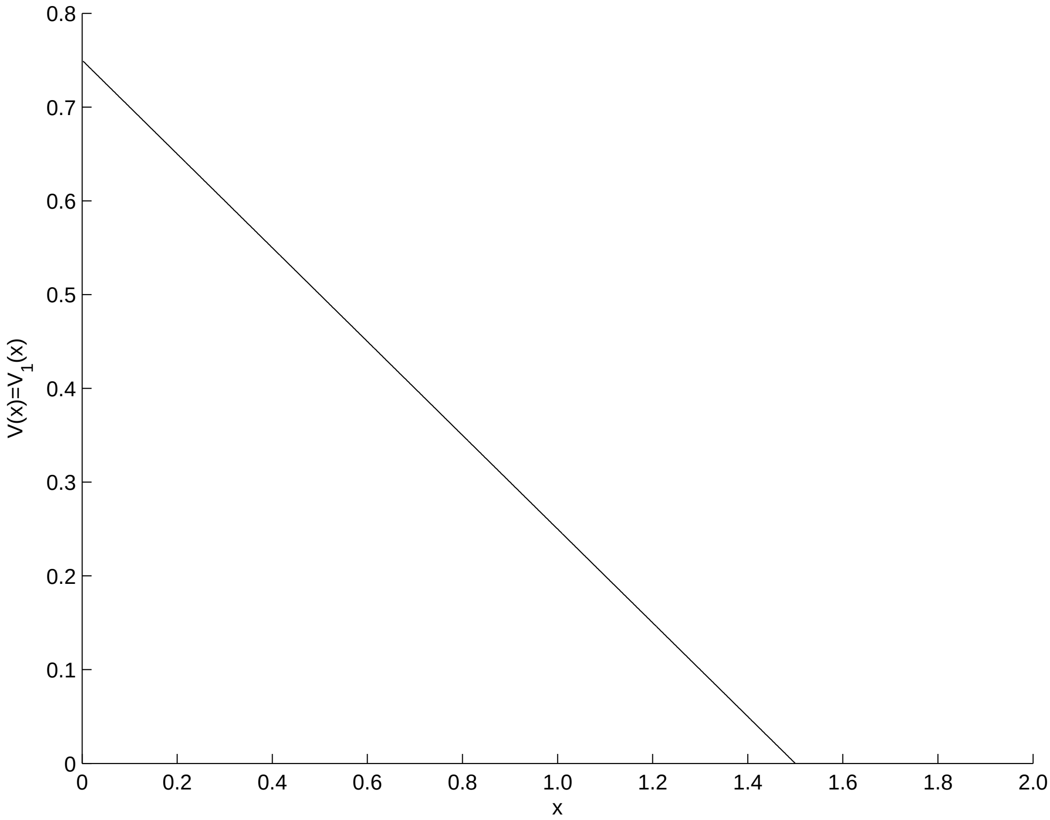

Figure 1: $p = 0.5$ and $\delta \leq 2/3$

3.2 Main Results.

Let us identify the region on δ and p plane, where the island will be dug in exactly one period. This region can be described by the following inequality

$$\Psi_1(t_1) \leq 0. \tag{51}$$

Using (12) and (28), inequality (51) can be rewritten as

$$t_1 \geq u_1, \tag{52}$$

or

$$\delta \leq \frac{2p}{1+p}. \tag{53}$$

Inequality (53) describes the region where the island will be dug in exactly one period. Figure 1 shows an example of the value function when $p = 0.5$ and $\delta \leq 2/3$.

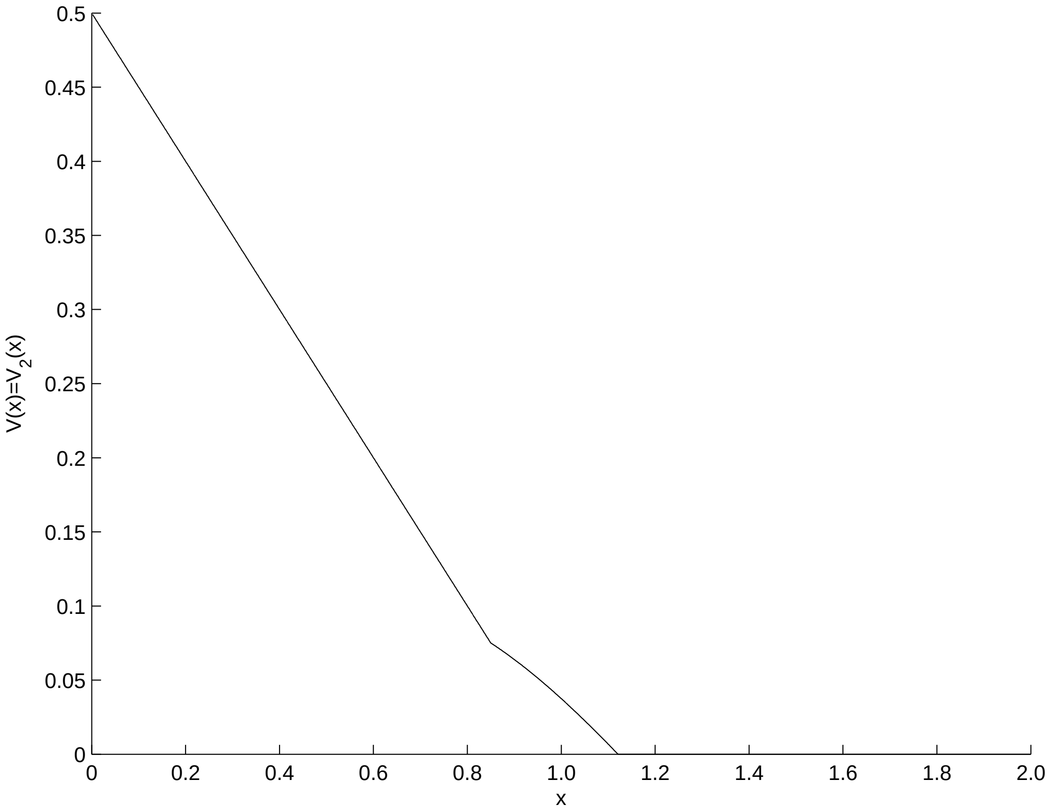

Figure 2: $p = 0$ and $\delta = 0.3$

The island will be dug in exactly two periods if the following inequality holds

$$t_2 \geq u_2. \tag{54}$$

Using Proposition 2, inequality (54) can be rewritten as

$$\frac{c_3 - c_2}{b_3 - b_2} + b_2 \geq \sqrt{b_2^2 + 2c_2}. \tag{55}$$

Inequalities (55) and the inverse of (53) describe the region where the island will be dug in exactly two periods. Figure 2 shows an example of the value function when $p = 0$ and $\delta = 0.3$.

DISCUSSION

Let us call $\Psi(x)$ the supremum of $\Psi_k(x)$ $k = 1, 2\ldots$ for every x and let us call $V(x)$ the corresponding value function. In Figure 4 we plot the value function when $\delta = 1$ and $p = 0$.

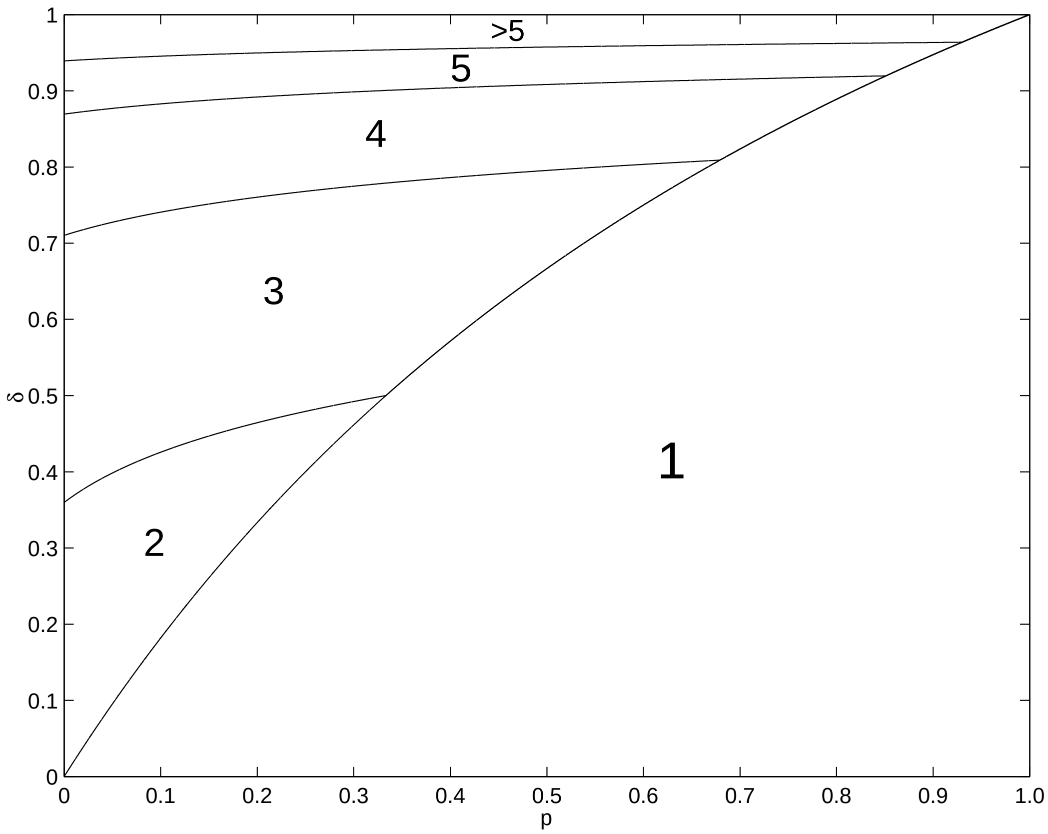

Figure 3: Amazing picture!

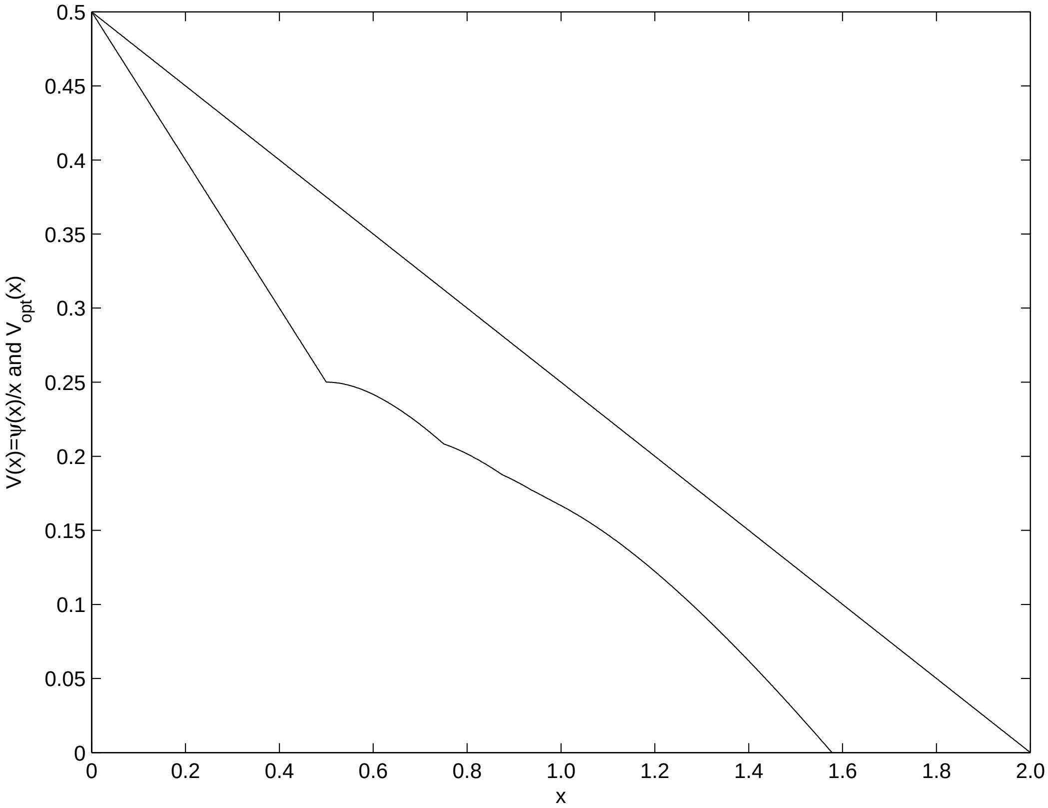

Figure 4: $V(x)$ and $V_{opt}(x)$

If the players behaved cooperatively then the best strategy for them would be to invest very tiny amount each period and finish the project in the infinitely large amount of time. The corresponding value function for one of the players when both players behave cooperatively is also plotted in Figure 4.

DISCUSSION

Now let us address the issue of how much cooperation can be supported for different parameter values. When can the project be only financed in one period and when say two period cooperation is feasible?

4 Extensions and Applications

A. General case: approximate by uniform step functions

B. n players

5 Related Literature

1. Contribution games: Phillipe Jehiel
2. R&D: $p < 1$: Jeniffer Reinganum
3. Stage financing: in finite or infinite number of periods
4. Public goods: $p = 1$
5. Multiple equilibria

6 Conclusion

In this paper, we analyze a dynamic search model. We develop new methods in order to analyze our simple setting. Our symmetry assumption makes the whole analysis much simpler and much more transparent.

We consider a situation when players are looking for the treasure. The treasure can be a public or private good, or it can have properties of both.

We show how players should invest.

We find that the discount factor and the degree of the good to be public are substitutes.

We show that the investment decision (a number of investment periods) is not continuous function of the discount factor for a fixed degree of the good to be public.

Applications: public goods, P&D, patent races

Discussion of the related literature.

Appendix

Proof of the lemma!!!!!!!!!!

Derivation of b_k

Given $b_k = \frac{b_{k-1}+p/\delta}{2}$ and $b_1 = \frac{1+p}{2}$ let us solve for b_k. First, we find a particular solution by assuming that b_k does not depend on k and derive $b_p = \frac{p}{\delta}$. Second, we find a complementary solution $b_c = C(\frac{1}{2})^k$. A general solution is the sum of particular and complementary solutions $b_k = \frac{p}{\delta} + C(\frac{1}{2})^k$. Next, we use the initial condition $b_1 = \frac{1+p}{2}$ to determine the value of $C = 1 + p - \frac{2p}{\delta}$. Finally, $b_k = \frac{1+p}{2^k} + \frac{p}{\delta}\left(1 - \frac{1}{2^{k-1}}\right)$.

Derivation of c_k

Given $c_k = \frac{\delta b_{k-1}^2 - p^2}{2\delta} + \delta c_{k-1}$ and $c_1 = 0$ let us solve for c_k. Notice that from (44) and (45) it follows that

$$\delta b_{k-1}^2 - p^2 = (\delta b_{k-1} + p)(\delta b_{k-1} - p) = (2\delta b_k)(\delta(1+p) - p/2)/2^{k-1}.$$

Let us introduce $d_k = \frac{c_k}{(1+p-p/2\delta)\delta^k}$ and rewrite the difference equation as

$$d_k = \frac{b_k}{(2\delta)^{k-1}} + d_{k-1}.$$

Substitute the initial condition $d_1 = 0$ to derive

$$d_k = d_1 + \sum_{i=2}^{k}(d_i - d_{i-1}) =$$

$$\sum_{i=2}^{k} \frac{b_i}{(2\delta)^{i-1}} = \sum_{i=2}^{k} \frac{2\delta(1+p-p/2\delta)}{(4\delta)^i} + \frac{2p}{(2\delta)^i}.$$

Next, given that $\sum_{i=2}^{k} \frac{1}{(2\delta)^i} = \frac{(2\delta)^{k-1}-1}{(2\delta-1)(2\delta)^k}$ and $\sum_{i=2}^{k} \frac{1}{(4\delta)^i} = \frac{(4\delta)^{k-1}-1}{(4\delta-1)(4\delta)^k}$ we have

$$c_k = (1+p-p/2\delta)\delta^k d_k =$$

$$2p(1+p-2p/\delta)\left(\frac{(2\delta)^{k-1}-1}{(2\delta-1)2^k}\right) + 2\delta(1+p-2p/\delta)^2\left(\frac{(4\delta)^{k-1}-1}{(4\delta-1)4^k}\right).$$

Solution when $\delta \to 1$

Let us calculate t_k and u_k when $\delta \to 1$. Note that $b_k \to p$ while $c_k \to \frac{(1-p)(1+5p)}{6}$ when $\delta \to 1$. This means that $u_k \to 1 + \sqrt{p^2 + \frac{(1-p)(1+5p)}{3}}$, while $t_k \to 1+p$ when $\delta \to 1$. Consequently, $t_k < u_k \ \forall \ k$ when $\delta \to 1$. Thus, when $\delta \to 1$ there is unbounded number of steps.

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