Assessing the turnpike hypothesis with the Rawls criterion

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Abstract

The Turnpike theorem for the model with the terminal criterion is one of the most important results in the linear theory of Neumann-Gale economic dynamics. Recently, however, considerable attention has been paid to a model with another criterion proposed by Rawls. To our knowledge, the 'Turnpike Hypothesis' (TH) for the model with the Rawls criterion has not been assessed yet. The present paper gives the answer to this question. By constructing examples of regular technologies for which trajectories do not converge, or converge to different rays, we show that the answer is negative. This work rejects not only the TH, but also the strong hypothesis. Descriptions of direct and conjugate (dual) models and corresponding mathematical apparatus are given. The paper also indicates that a related paper Kaganovich (2000) - which shows that under the Rawls criterion there is a rolling-plan mechanism converging to the turnpike - contains an error.

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1 Introduction

Investigation of infinite time horizon *stationary* models (in which technology is constant with time) is one of the most methodologically important fields in the linear theory of Neumann-Gale economic dynamics.[1] This field adopts two alternative approaches to the notion of optimality.

The first approach considers a closed system. The technological description of such a system comprises reproduction of all resources including labor. Consumption under this approach has a purely *internal* productive role and is included in the description of the technology. This economic system is said to be *closed with respect to consumption*; it does not have any external goals. Rather, its natural internal goal is development at the maximal rate. This is a mostly abstract and simple model, but it allows the introduction of such fundamental notions as equilibrium, the von Neumann ray of maximal balanced growth, the Bellman operator, and potential.

The second approach considers consumption separately. Consumption is described by an exogenously given utility function U and does not influence the technological capabilities of the economic system. This system is said to be *open with respect to consumption*. Models of the open type are divided into two groups; classification depends on how labor is included within the models.

In the first group of models with an open type of economic system, labor is not a limiting factor, therefore labor is not included in the technology description. The usual criterion for such a model is maximization of the discounted sum of utilities:

$$\varphi := \sum_{t=0}^{\infty} \mu^t U(c_t) \longrightarrow \max . \tag{1}$$

In the second group of models with an open type of economic system, labor resources are an additional constraint on technological growth. In these models the growth rate of the economic system is determined by the exogenously given growth rate of labor L_t. Maximization of average consumption per capita during the planning

[1]See, for example, [3, 5, 13, 19, 20, 21, 26].

period T is a usual criterion in this framework:

$$\varphi := \frac{1}{T}\sum_{t=0}^{T} U\left(\frac{c_t}{L_t}\right) \longrightarrow \max . \tag{2}$$

One of the most important theoretical results here is the 'Consumption Turnpike Theorem'. Nikaido (1968, ch. 14) outlines a complete version of this theorem, originally obtained by Atsumi, Gale and Tsukui. The theorem states that with large T, the optimal path issuing from any initial state approximates the 'golden ray', on which a trajectory has balanced growth with a maximal level of consumption.

Sheinkman (1976) and Cass and Shell (1976) were the first to consider the discounted sum of utilities criterion (1) instead of average criterion (2) for these models. They obtained results similar to the 'Consumption Turnpike Theorem', but only for discount factors sufficiently close to 1 and for differentiable functions. More recently, Montrucchio (1995) proves the asymptotic local turnpike without differentiability assumptions, but this paper imposes some mild curvature restrictions on the utility function.

Nowadays considerable attention has been paid to open models with another criterion proposed by Rawls (1972). In short, the Rawls principle can be formulated in the following way: all the generations should have the same welfare. To incorporate this concept, a number of studies (see, for example, [1, 2, 6, 11, 12, 16]) have examined the optimal paths in the Neumann-Gale linear theory under the following maximin criterion:

$$\varphi := \min_{t\geq 0} U\left(\frac{c_t}{L_t}\right) \longrightarrow \max . \tag{3}$$

This criterion requires the minimum level of per capita utility among generations to be maximized.[2]

However, to our knowledge there is no systematic analysis in the literature of asymptotic behavior of optimal trajectories under the Rawls criterion. For instance, Kaganovich (2000) considered only a basic case with Leontief technology. He showed

[2]For a discussion on the importance of the criterion, see Solow (1992) and Hammond (1993).

that in this case there is a golden ray similar to Nikaido's. In [16] Kaganovich tried to find an answer to the different question of whether a rolling-plan mechanism converges to a golden ray; indeed, the main result of the paper is that this is the case with Leontief technology.

The main goal of this paper is to provide a systematic analysis of asymptotic behavior of optimal trajectories under the Rawls criterion. Many of the concepts and results in our paper have appeared earlier in Russian but appear to be new in English language literature. For example, our work [6] develops mathematical tools based on the theory of conjugate functions; this allows the analysis of linear models in a general form. In that paper we formulate the previously unresolved question whether, for regular Gale technology (see definition 1 below), the Rawls model always has a turnpike similar to the one found by Nikaido. Our work [9] answers this question in the negative. We construct examples of regular technologies (of a simple, rectangular type) for which trajectories do not converge, or converge to different rays.

In the present paper we also comment on Kaganovich's results. In particular, we show that [16] contains an error, and the question whether under the Rawls criterion a rolling-plan mechanism converges to a golden ray remains open (even in the case of Leontief technology).

The rest of the paper is organized as follows. The main concepts of a closed regular Gale technology are formally introduced in Section 2. Section 3 constructs a general open model with the Rawls criterion; for this model, the solution in terms of the Bellman equation is given and the turnpike hypothesis is formulated. In Section 4 mathematical tools using conjugate functions are developed and the general solution for the Rawls model in conjugate space is given. Section 5 constructs counter-examples to the turnpike hypothesis. Section 6 is devoted to the discussion of the results of this paper. It also comments on [16].

2 Closed Regular Gale Technology

Gale technology is defined by a point-to-set map of phase space $X := R^n_+$ into X itself, $\omega : X \to X$. State $x \in X$ is interpreted as a vector of initial resources (products); the points y of the set $\omega(x) \subset X$ are possible production plans from resources x at the unit of time. Thus, the transition map $x \to y \in \omega(x)$ describes feasible dynamics of the system in discrete time.

Note 1. *Let us point out that labor is included in the list of resources (products). It is considered to be a factor of production. In other words, the described Neumann-Gale model is closed in consumption: all resources needed for production (including labor) are taken into account, hence no additional input is required.*

The following properties of technology ω are assumed:

a) for every $x \in X$ the set $\omega(x)$ is a nonempty convex compact;

b) with every point y the set $\omega(x)$ contains the whole cone segment $[0, y]$[3] (the principle of free disposal);

c) homogeneity of degree one: $\omega(\lambda x) = \lambda\omega(x) \quad \forall\, x \in X$ and $\lambda \geq 0$;

d) superadditivity: $\omega(x_1 + x_2) \supseteq \omega(x_1) + \omega(x_2)$;

e) nontriviality: for any strictly positive x ($x > \mathbf{0}$) the set $\omega(x)$ contains a strictly positive vector y ($y > \mathbf{0}$).

Note that from b) and d) it follows that the map ω is monotonic, i.e. $\omega(x_1) \subseteq \omega(x_2)$ if $x_1 \leq x_2$. Moreover, Krass (1971) proved that the map ω has to be continuous.

Such a technology ω is termed *Gale technology.* The properties of Gale technology (see, for example, Karlin 1959, section 9.10) are usually formulated in terms of technological cone Z, which is the graph of map ω in space $R^{2n}_+ = X * X$, that is

$$Z := \{(x, y) \in X * X \mid y \in \omega(x)\}.$$

Z is a closed, convex cone.

[3]The paper uses the standard notation: for $x, y \in X$ inequality $x \leq y$ ($x < y$) means that all components of x are not more (strictly less) than components of y; for $x \leq y$ by cone segment $[x, y]$ we call the rectangle set $\{z \in X \mid x \leq z \leq y\}$.

Map ω defines sets, reachable by one step. By iterating ω, we can derive maps ω^k that define sets reachable by k steps:

$$\omega^0(x) := \{x\},\ \ \omega^{k+1}(x) := \{y \in \omega(z) \mid z \in \omega^k(x)\} \quad x \in X \quad k = 0, 1, ... \quad .$$

A trajectory is defined as an infinite sequence $\{x_t,\ t = 0, 1, ...\}$ issuing from some point $x \in X$, generated by relations

$$x_0 = x,\ \ x_{t+1} \in \omega(x_t) \quad t \geq 0 \quad . \tag{1}$$

A trajectory is called *effective* if

$$x_t \in \Pi(\omega^t(x)) \quad \forall \quad t = 0, 1, ... \ ,$$

where $\Pi(Q)$ is a *Pareto boundary* of convex compact $Q \subset X$:

$$\Pi(Q) := \{x \in Q \mid x \leq y \in Q \Longrightarrow y = x\} \quad . \tag{2}$$

2.1 Regular technology

The main results of the classical turnpike theory give a description of an ultimate attractor, i.e. the smallest set (a cone in R^n_+) that attracts effective trajectories from any initial point $x \in X$. Strong turnpike theorems characterize conditions where an attractor is a ray

$$R^+h := \{x = \lambda h \mid \ \lambda \geq 0\} \quad , \tag{3}$$

where $h \in X$ is a some vector. This ray is called a Neumann-ray (a *turnpike*). Our paper considers exactly this case, which will be described in terms of an equilibrium.

For any Gale technology ω there is an *equilibrium*: a triplet (α, h, π), $\alpha > 0$, $h \in X$ and $\pi \in X^*$ ($X^* = R^n_+$ is the space conjugate to X) connected by relationships

$$\text{a) } \alpha h \in \omega(h) \quad \text{and} \quad \text{b) } \pi y \leq \alpha \pi x \quad \forall \quad (x, y) \in Z \quad . \tag{4}$$

Vector h and constant $\alpha > 0$ define the Neumann ray (3) of balanced growth at maximal rate; components of vector π are equilibrium prices. Generally the equilibrium can be nonunique or trivial. The following definition specifies the case the paper deals with.

Definition 1. *Gale technology ω is termed regular if:*

1) triplet (α, h, π) satisfying relations (4) is unique (naturally, vectors h and π are accurate to within a scalar);

2) $h > \mathbf{0}$, $\pi > \mathbf{0}$.

It follows from condition $\pi > \mathbf{0}$ that simplex

$$\sigma := \{x \in X \mid \pi x = 1\} \tag{5}$$

is a compact set. In addition, assume that equilibrium vectors h and π are connected by the following condition

$$\pi h = 1, \text{ that is } h \in \sigma. \tag{6}$$

For a regular Gale technology the ultimate attractor is a ray (3); we denote this ray by Δ.

The conditions of Definition 1 are relatively strong. However, from an economic point of view a regular technology corresponds to the general case of a nontrivial economic system. Let us consider the following example.

Example 1. *Leontief technology is one of the simplest but most economically important examples. It is given by the map*

$$\omega(x) := \{y \in X \mid Ay \leq x\} \quad x \in X \quad , \tag{7}$$

where $A(n * n)$ is a nonnegative quadratic input coefficients matrix. To assure the regularity of Leontief technology it is enough to assume strict positivity of all elements *of matrix A. The equilibrium triplet of this model is described by spectral radius* $r = r(A)$ *and left and right eigenvectors:*

$$\alpha = \frac{1}{r}, \;\; Ah = rh \;\text{ and }\; \pi A = r\pi.$$

When $r < 1$ $(\alpha > 1)$ *the economy is called productive, see Karlin (1959, ch. 9).*

2.2 Space of feasible functions, Bellman operator and potential

Working with optimization problems we will use feasible functions as described in Definition 2.

Definition 2. *A function* $\Phi : R^n_+ \to R_+$ *is feasible if the following conditions are met:*

a) monotonicity: $\Phi(x_1) \leq \Phi(x_2)$, *if* $x_1 \leq x_2$;

b) homogeneity of degree one: $\Phi(\lambda x) = \lambda\Phi(x)$, $\forall$ $(x \in R^n_+,\ \lambda \geq 0)$;

c) continuity: Φ *is continuous on* R^n_+.

The set of feasible functions is denoted by W*; by* $\hat{W}$ *and* $\check{W}$ *we mean subsets of the feasible functions that are concave and convex respectively.*

Note 2. *It was shown in [3] that conditions (a) and (b) ensure the continuity of* Φ *for all internal points in* R^n_+ $(x \in intR^n_+)$. *Condition (c) requires continuity in the whole* R^n_+, *including the boundary* ∂R^n_+.

Now, let us assume that there is a natural ordering in W

$$\Phi_1 \preceq \Phi_2 \quad \Leftrightarrow \quad \Phi_1(x) \leq \Phi_2(x) \quad \forall\ x \in X \quad . \tag{8}$$

The compactness of simplex (5) allows the introduction of the following metric in W

$$\rho(\Phi_1, \Phi_2) := \max_{x \in X} \frac{\mid \Phi_1(x) - \Phi_2(x) \mid}{\pi x} = \max_{x \in \sigma} \mid \Phi_1(x) - \Phi_2(x) \mid . \tag{9}$$

Using this metric we define norm $\|\Phi\| := \rho(\Phi, \Phi_2 \equiv 0)$ and the notion of convergence in W. In metric (9), W is a complete metric space.

Definition 3. *By the Bellman operator of technology* ω *we call operator* $\Gamma : W \to W$ *given by formula*

$$\Gamma\Phi(x) := \max_{y\in\omega(x)} \Phi(y) \quad x \in X, \ \ \Phi \in W. \quad \square \tag{10}$$

Using the inequality

$$| \, \Gamma\Phi_1(x) - \Gamma\Phi_2(x) \, |\leq \max_{y\in\omega(x)} | \, \Phi_1(y) - \Phi_2(y) \leq$$

$$\leq \rho(\Phi_1, \Phi_2) \cdot \max_{y\in\omega(x)} \pi y \leq \rho(\Phi_1, \Phi_2) \cdot \alpha\pi x$$

the following lemma can be proved.[4]

Lemma 1. *The Bellman operator* Γ *is monotonic in the sense of (8); due to the superadditivity of map* ω *subspace* $\hat{W}$ *is invariant, i.e.* $\Gamma\Phi \in \hat{W}$ *if* $\Phi \in \hat{W}$*. Besides, the following inequality holds*

$$\rho(\Gamma\Phi_1, \Gamma\Phi_2) \leq \alpha\rho(\Phi_1, \Phi_2) \quad , \tag{11}$$

in particular,

$$\|\Gamma\Phi\| \leq \alpha\|\Phi\| \quad \forall \ \Phi \in W,$$

and consequently Γ *is bounded and continuous.*

Corollary 1. *For any* $\delta > \alpha$ *the operator* $\delta^{-1}\Gamma$ *is a contraction operator in metric (9).*

2.3 Turnpike theorem for an optimization problem with a terminal functional

Lemma 2. *([26, ch. 15], [4]). If technology* ω *is regular, then:*

[4] Also see (9) and (4,b).

a) Operator Γ *has only one eigenvalue, equal to* α*. Its normalized eigenfunction*

$$G: \ \ \Gamma G = \alpha G \qquad G(h) = 1, \tag{12}$$

is called a potential, and is unique and concave, $G \in \hat{W}$*;*

b) For any function $\psi \in W$*, sequence*

$$V_0 := \psi, \ \ V_k := \alpha^{-1} \Gamma V_{k-1} \ \ k = 1, 2, ... \tag{13}$$

converges. Furthermore, the limit of this sequence coincides with potential G, multiplied by some scalar, which depends on ψ*.*

Let us consider the optimization problem with a terminal functional as a criterion (this problem is natural for a model closed in consumption, see Note 1):

$$\alpha^{-T} \psi(y) \longrightarrow \max_{y \in \omega^T(x)} =: V_T(x) \quad . \tag{14}$$

Here, T is interpreted as a planning horizon and the resulting functional V_T as a *value function* of horizon T. Applying dynamic programming recursive methods we deduce:

1) functional (14) is an element of sequence (13) with $k = T$;

2) since the optimal trajectory of problem (14) is constructed by the rule

$$x_0 = x, \ \ x_{t+1} = \operatorname*{argmax}_{y \in \omega(x_t)} V_{T-t}(y) \quad t = 0, 1, ..., T-1$$

from part b) of Lemma 2, it follows that the optimal trajectory as $T \longrightarrow \infty$ converges to an infinite effective trajectory determined by the formula

$$x_0 = x, \ \ x_{t+1} = \operatorname*{argmax}_{y \in \omega(x_t)} G(y) \quad t = 0, 1, ... \quad , \tag{15}$$

and is independent of functional ψ of problem (14). This means that potential G is a *generator of infinite effective trajectories*. Hence, the following theorem holds.

Theorem 1. *([26,3,21,5]). For a regular Gale technology the following statements* *are valid:*

1) Infinite effective trajectories (which are called optimal) are generated by the potential transition (point-to-set) map:

$$x \longrightarrow Y(x) := \operatorname*{Argmax}_{y \in \omega(x)} G(y) \quad , \tag{16}$$

i.e. for every initial point $x_0 \in X$ such a trajectory is given by (15);

2) On an effective trajectory the potential grows with time exactly as a geometric progression with rate α:

$$G(x_t) = \alpha^t G(x_0) \quad \forall \ t = 0, 1, ... \quad ; \tag{17}$$

3) A discounted trajectory $\{\xi_t := \alpha^{-t} x_t\}$ converges to a point on the turnpike Δ with a coordinate equal to the potential value of the initial point:

$$\xi_t \longrightarrow \xi^* := G(x_0) * h \in \Delta \quad t \longrightarrow \infty \quad . \tag{18}$$

Relation (18) is actually the turnpike theorem for infinite time trajectories with regular technology, formulated in the terms of the present paper. This relation explains why function G is termed 'potential'. Together with triplet (4), the potential forms the *characteristic quadruple* (α, h, π, G) of regular technology ω.

Example 2. *(continuation of example 1, see section 2.1). The potential of Leontief technology (7) is the complement function (L. Kantorovich's term) with etalon vector $h: \ G = \Lambda_h$, where for any fixed etalon vector $a = (a_i) \in R^n_+ \quad (a \neq \mathbf{0})$*

$$\Lambda_a(x) := \min_{1 \le i \le n} \frac{x_i}{a_i} = \max\{\lambda \mid \lambda a \le x\} \ \ x = (x_1, ..., x_n) \in X \quad . \tag{19}$$

Let us show that the condition of equation (12) is satisfied for the function Λ_h. The effective trajectory (15) in this case is

$$x_0 = x, \ \ x_t = \alpha^t G(x_0) * h \ \ t = 1, 2... \tag{20}$$

It means that the trajectory enters the turnpike Δ by one step and further moves along it at rate α. In discounted form, ξ_t is equal to ξ^* for all $t \geq 1$.

3 Open model with external consumption

The material of this section mainly follows our work [6].

3.1 Model

In the closed model described in the previous section, let us introduce two exogenous factors - population and external consumption.

As was mentioned in section 2.1 (Note 1), labor is included in the description of technology ω as one of the input factors and, on the other hand, as one of the products. The notion of population has a wider meaning as it includes both workers and non-workers. Now assume that population grows at a constant rate λ. In units of the initial population

$$L_t = \lambda^t \ , \ \ \lambda = \text{const} > \mathbf{0} \quad . \tag{1}$$

Note that L_t is measured as the *actual* physical volume of population (number of consumers). By contrast, labor is measured in units that also take into consideration labor productivity; labor can grow faster by means of improving working skills. On the optimal trajectory of technology ω, labor can grow at rate α (together with all other resources).

The description of technology ω in section 2.1 included consumption induced by reproduction of labor resources; this consumption is called *productive* consumption, and is needed only for purposes of production. However, the population, including the labor force, also needs nonproductive consumption such as luxuries, travel, sport, and hobbies. Non-productive consumption c is outside the technology used in this model, therefore we call it *external consumption.* It receives, however, resources from the same sources as productive consumption, so that $c \in R^n_+$, $\mathbf{0} \leq c \leq x$, where x

is a vector describing the current state of the economic system. The utility function of such consumption (denote it by U) represents the pleasures of human life. We assume, that

U1. utility function U is feasible and concave, $U \in \hat{W}$.

Thus, now in an *open* model, the system trajectory is a sequence of pairs $\zeta := \{(x_t, c_t),\ t = 0, 1, ...\}$, connected by relations:

$$x_0 = x, \ \ \mathbf{0} \le c_t \le x_t, \quad x_{t+1} \in \omega(x_t - c_t) \quad t = 0, 1... \quad . \tag{2}$$

Note 3. *Recall that our interpretation of the open model is different from the traditional one, as described in the introduction. We added to the traditional closed model a utility function of external consumption.*

Another external parameter that we would like to introduce into the model is a rate of time preference ρ. This parameter ρ is given exogenously and has a *normative* character. In contrast to the Rawls criterion (1.3), we will assess trajectory ζ by value

$$\varphi(\zeta) := \min_{t \ge 0} \frac{1}{\rho^t} * U\left(\frac{c_t}{\lambda^t}\right) = \min_{t \ge 0} \frac{U(c_t)}{\delta^t} \quad \delta := \rho\lambda. \tag{3}$$

Note 4. *The pure Rawls criterion (1.3) can be obtained from (3) by substitution $\rho := 1$ and $\delta := \lambda$. However, there is an in-principle difference between parameters λ and δ. While λ is determined only by the historic process, and so has a real character, parameter ρ, and consequently δ, can be regulated and altered by the government. Note that a similar parameter was introduced independently in Kaganovich (2000).*

Thus, the model described above is characterized by the triplet $\{Z\ \ (or\ \ \omega), U, \delta\}$. Using $Tr(x)$ to denote *the bundle of trajectories* ζ starting from the initial point $x \in X$ (ζ is defined in (2)), we introduce the following problem:

$$\varphi(\zeta) \longrightarrow \max_{\zeta \in Tr(x)} =: I(x) \qquad x \in X \quad . \tag{4}$$

3.2 Bellman equation, Theorem 2

The value function I introduced in (4) is a solution of stationary (constant with time) Bellman equation

$$I: \qquad \Phi(x) = \max_{\mathbf{0}\le c\le x} \min \left[U(c), \frac{1}{\delta} * \Gamma\Phi(x-c) \right] \quad x \in X, \ \ \Phi \in W \quad , \tag{5}$$

where Γ is a Bellman operator (2.10) of technology ω.

Equation (5) is the main equation of this model. Its solution determines for every state $x \in X$, the utility level $I(x)$, the optimal consumption vector $c = C(x)$, maximizing the right side of (5) under $\Phi = I$, and the transition map

$$x \longrightarrow Y(x) := \underset{y\in\omega(x-C(x))}{\operatorname{argmax}} \ I(y) \quad . \tag{6}$$

An optimal trajectory is constructed using the following rule

$$x_0 = x, \ \ c_t := C(x_t), \ \ x_{t+1} := Y(x_t) \quad t = 0, 1, ... \quad . \tag{7}$$

Equation (5) always has a trivial solution, $\Phi \equiv 0$. The existence of nontrivial solutions is clarified by the following theorem.

Theorem 2. *([6]). In a regular model with $U \not\equiv 0$:*

1) a nontrivial solution $I \not\equiv 0$ exists (and is unique) iff $\delta < \alpha$, where α is the von Neumann growth rate of technology ω;

2) function I can be constructed as the limit of the monotonically decreasing iterative sequence

$$\Phi_0 = \infty, \ \ \Phi_{k+1} = J\Phi_k \quad k = 0, 1, ... \ , \tag{8}$$

where J is the operator of the right side of (5);

3) optimal trajectory is uniformly balanced, that is

$$I(x_t) = U(c_t) = \delta^t I(x_0) \quad \forall \ t = 0, 1, \ldots \tag{9}$$

consequently, utility per capita is equal to

$$U(c_t)/\lambda^t = \rho^t I(x_0)$$

and grows with time strictly in accordance with the rate ρ.

Proof. We will prove the first statement in subsection 4.4. To do that we will need the tools of conjugate functions that will be developed in section 4. If the first statement holds, the second statement follows from the monotonicity of operator J with respect to function Φ (in the sense of (2.8)). The third statement follows from the Corollary of Lemma 4 (see subsection 4.3 below). □

Note 5. *It follows from (1) and (3) that consumption per capita can actually increase* ($\rho > 1$) *only if* $\lambda < \alpha$, *that is, only if the growth rate of population is less than the growth rate of technology.*

3.3 Normative growth rate and the turnpike hypothesis

Relation (9) shows that the growth rate of the optimal trajectory is now equal to $\delta = \lambda\rho$. This is in contrast to the result of the closed model of section 2 in which there is no external nonproductive consumption and the optimal trajectories grow with the maximal possible technological growth rate α. This raises two important questions:

1) Is there any ray of stationary normative growth? Specifically, is there a ray (denoted by $l \subset X$) that for $x_0 \in l$ the optimal trajectory (7) is

$$x_t = \delta^t x_0 \in l \quad \forall \quad t \quad ? \tag{10}$$

2) Will l be a turnpike? That is, will l attract optimal trajectories (7) from all other initial points? This question is called the *Turnpike Hypothesis.*

Ray l, defined in (10), exists iff there is a pair (x^*, c^*) such that

$$I(x^*) = U(c^*) \text{ and } \delta x^* \in \omega(x^* - c^*) \quad . \tag{11}$$

If this holds, then $l = R_+ x^*$. In Nikaido's well-known 'Consumption Turnpike Theorem', such a pair is called the golden pair (see Nikaido [23, section 14.2]). In [6] it was shown that in the Rawls model the golden pair (11) exists and, moreover, the hypothesis that the corresponding ray is a turnpike was formulated.

Example 3. *(continuation of examples 1 and 2, see sections 2.1, 2.3). Let us use the Leontief technology again. As utility function U, we take a complement function Λ_b (see 2.19) with some etalon vector $b \in X$. In this case I is also a complement function Λ_d with etalon*

$$d = (E - \delta A)^{-1} b \quad ,$$

where E is the unit matrix. The optimal trajectory is described by the following equation:

$$(x_t, c_t) = \delta^t I(x_0) * (d, b) \quad t \geq 1 \quad .$$

It means that the pair (d, b) is a golden pair. The optimal trajectory enters the turnpike immediately by one step, see [7].

In the general case, however, the turnpike hypothesis is not valid. Indeed, the main result of this paper is that the turnpike hypothesis does not hold even with simple models, for example with rectangular technology.

4 Conjugate technology and tools based on conjugate functions

This section gives a short description of conjugate theory for the space of functions $\hat{W}$. A full version of this theory can be found in [8].

4.1 Operation of conjugation

Conjugate functions are defined on the space of prices $p \in X^* = R^n_+$ dual to the space of products X. The components of a *vector* p are interpreted as prices of the resources (products).

In the space of feasible functions W the operation of conjugation is defined not in additive form (as in the Legendre-Fenchel transformation, see [25, §12]), but in multiplicative form. For a concave function $\Phi \in \hat{W}$ this operation is defined by the following formula:

$$\Phi^*(p) := \min_{x \in X} \frac{px}{\Phi(x)} = \min_{x \in D(\Phi)} px \quad p \in X^* \tag{1}$$

where

$$D(\Phi) := \{x \in X \mid \Phi(x) \geq 1\} \;\; \Phi \in \hat{W}. \tag{2}$$

The main property of conjugation is established in the following lemma.

Lemma 3. *([25, §15], [4, section 1.7]). For any arbitrary $\Phi \in \hat{W}$ the following properties always hold*

$$\text{a) } \Phi^* \in \hat{W} \qquad \text{b) } \Phi^{**} = \Phi \quad .$$

This means functions Φ and Φ^* form a *conjugate pair*, that is they are dual to each other.

The following example will be used later in the paper: a complement function Λ_a is conjugate to a linear function $a(p) = pa$, that is

$$\Lambda_a^*(p) = \min_{x} \frac{px}{\Lambda_a(x)} = pa \quad p \in X^* \quad . \tag{3}$$

4.2 Supergradient and principle of reciprocity

A set of vectors x, on which the minimum in (1) is reached, is called *superdifferential* of function Φ^* at point p (see [25, §23]), that is

$$\mathrm{Grad}\Phi^*(p) := \operatorname*{Argmin}_{x \in D(\Phi)} px = \{x \in X \mid \Phi(x) = 1, \;\; px = \Phi^*(p)\} \subset X \tag{4}$$

where $D(\Phi)$ is defined in (2). Each of these vectors is called a *supergradient* and denoted by $\mathrm{grad}\Phi^*(p)$. The map $p \to \mathrm{Grad}\Phi^*(p)$ is homogeneous of degree zero. If Φ^* is differentiable at point p then set (4) consists only of one vector, which is the gradient in the usual sense.

Using Lemma 3 we similarly introduce

$$x \to \mathrm{Grad}\Phi(x) := \operatorname*{Argmin}_{p \in D(\Phi^*)} px = \{p \in X^* \mid \Phi^*(p) = 1,\ \ px = \Phi(x)\} \subset X^*. \qquad (5)$$

The above relationships are equivalent to

$$p \sim \mathrm{grad}\Phi(x) \quad \Leftrightarrow \quad x \sim \mathrm{grad}\Phi^*(p) \quad \Leftrightarrow \quad px = \Phi^*(p)\Phi(x) \quad , \qquad (6)$$

where $\sim$ means that the vectors are proportional to each other. A pair of vectors (p, x) satisfying relationship (6) is termed Φ-*reciprocal*.

4.3 Operator of uniform sharing

Let us introduce operator $S : \hat{W} * \hat{W} \longrightarrow \hat{W}$

$$S: \qquad S[\Phi_1, \Phi_2](x) := \max_{\substack{x_1, x_2 \in X \\ x_1 + x_2 \leq x}} \min[\Phi_1(x_1), \Phi_2(x_2)] \qquad x \in X. \qquad (7)$$

Operator S has the interpretation of a sharing operator. If Φ_1, Φ_2 are utility functions of two individuals and x is a vector of resources, then x_1, x_2 are shares to individuals 1 and 2 respectively.

Lemma 4. *([6], the proof is in the Appendix).*

1) The following equalities are valid

$$\text{a)}\ \ S^* = \Phi_1^* + \Phi_2^* \qquad \text{b)}\ \ \mathrm{Grad}\ \ S^* = \mathrm{Grad}\ \ \Phi_1^* + \mathrm{Grad}\ \ \Phi_2^* \quad . \qquad (8)$$

Corollary 2. *For every $x \in X$ there is always a balanced sharing, that is a pair x_1, x_2, such that*

$$x_1 + x_2 = x, \quad \Phi_1(x_1) = \Phi_2(x_2) = S(x).$$

In addition, if vector p is S-reciprocal with x, then p is Φ_i-reciprocal with x_i $i = 1, 2$.

Proof. It is sufficient to prove this statement for $x \in D(S)$. Let p be an S-reciprocal to x, then $x \in \mathrm{Grad}S^*(p)$. From (8,b) it follows that we can divide x into $x_1 + x_2$ where $x_1 \in \mathrm{Grad}\Phi_1^*(p)$, $x_2 \in \mathrm{Grad}\Phi_2^*(p)$. From (5) we have $\Phi_i(x_i) = 1$, therefore the sharing $x = x_1 + x_2$ is the balanced sharing. □

4.4 Conjugate technology and conjugate equation

Conjugate to technology cone (2.1) is the following cone (see [20]):

$$Z^* := \{(p, q) \in X^* * X^* \mid qy \le px \ \forall \ (x, y) \in Z\} \subset^* R_+^{2n} \ . \tag{9}$$

The inequality in (9) means that the value of the goods is no more than the value of the resources from which these goods were produced.

Let us denote by ω^* and Γ^* conjugate technology and its Bellman operator respectively. Note that if ω is regular then ω^* is a regular Gale technology as well. The following lemma can be proved directly.

Lemma 5. *The following relations are valid:*

$$\text{a)}\ \Gamma^*\Phi^* = (\Gamma\Phi)^* \quad \forall \ \Phi \in \hat{W}; \qquad \text{b)}\ (\alpha, h, \pi, G)^* = (\alpha^{-1}, \pi, h, G^*) \quad . \tag{10}$$

The characteristic quadruple of conjugate technology is presented on the left side of b).

Corollary 3. *From (10), Definition 1 and the principle of reciprocity it follows that technology ω is regular iff technology ω^* is regular.*

Example 4. *(continuation to examples 1-3, see sections 2.1, 2.3, 3.3). Conjugate to Leontief technology (2.7) is rectangular technology*

$$\omega^*(p) = \{q \in X^* \mid q \le pA\} \quad p \in X^*. \tag{11}$$

This technology is termed rectangular because region $\mathbf{0} \le q \le pA$ is a rectangle (see footnote 2). From (9,b), the growth rate of this technology is $\alpha^{-1} = r(A)$ and the potential is (see (3))

$$G^*(p) = \Lambda_h^*(p) = ph \quad p \in X^* \quad . \tag{12}$$

Using the definition of the sharing operator (7), equation (3.5) can be rewritten in the following way

$$\Phi = S[U, \delta^{-1}\Gamma\Phi] \quad . \tag{13}$$

Using (8,a) we can write

$$\Phi^* = U^* + (\delta^{-1}\Gamma\Phi)^* = U^* + \delta(\Gamma\Phi)^* \quad ,$$

and due to (10,a) we derive the Bellman equation in dual space

$$I^* : \qquad \Phi^* = U^* + \delta\Gamma^*\Phi^* \qquad \Phi^* \in \hat{W} \quad . \tag{14}$$

Define $\gamma := \delta\alpha^{-1}$. The following theorem is dual to Theorem 2 in subsection 3.2.

Theorem 3. *In a regular model when $U \not\equiv 0$*

1) a solution of equation (14) I^* exists (and is unique) iff $\gamma < 1$;

2) function I^* can be derived as a limit of the monotonically increasing iterative sequence

$$\Phi_0^* \equiv 0, \;\; \Phi_{k+1}^* = J^*\Phi_k^* \quad k = 0, 1, \ldots \;, \tag{15}$$

where J^* is the operator of the right side of (14),

3) the sequence (15) is conjugate to sequence (3.8).

Proof. When $U \not\equiv 0$ from (14), it follows

$$\Phi^*(p) > \delta\Gamma\Phi^*(p) \qquad \forall p > \mathbf{0} \quad . \tag{16}$$

Applying condition (2.4,a) to technology ω^*, we have

$$\Gamma^*\Phi^*(\pi) = \max_{q\in\omega^*(\pi)} \Phi^*(q) \geq \Phi^*(\alpha^{-1}\pi) = \ a^{-1}\Phi(\pi) \quad .$$

Together with (16) (for $p := \pi$), this leads to inequality $\Phi^*(\pi) > \gamma\Phi^*(\pi)$. This inequality leads to a contradiction if $\gamma \geq 1$.

If $\gamma < 1$ both operator J^* and operator Γ are contraction operators (see Corollary from Lemma 1 applying to ω^*). From a general property of a contraction map, equation (14) has a unique solution. This solution can be constructed by the method of iterations, in particular, as the limit of sequence (15). Note that the reciprocity of (15) and (3.8) follows from (8,a). □

Note that statement 1) in Theorem 3 leads to an analogous statement in Theorem 2.

Corollary 4. *If we rewrite equation (14) in the following form*

$$(E - \delta\Gamma^*)\Phi^* = U^* \qquad \Phi^* \in \hat{W}$$

then when $\gamma < 1$ operator $(E - \delta\Gamma^*)^{-1}$ exists and

$$I^* = (E - \delta\Gamma^*)^{-1}U^* \quad . \tag{17}$$

4.5 Reciprocity of trajectories of direct and dual problems, Theorem 4

According to (14)

$$I^*(p) = U^*(p) + I^*(q) \qquad q \in Q(p) \quad ,$$

where the transition map

$$p \longrightarrow Q(p) := \operatorname*{Argmax}_{q\in\omega^*(p)} I^*(q) \tag{18}$$

is induced by function I^*.

This means, I^* generates optimal trajectories for the conjugate Bellman equation (14):

$$p_0 = p \quad , \quad p_{t+1} := Q(p_t) \quad t = 0, 1, \dots \quad . \tag{19}$$

It is easy to see that equation (14) is the Bellman equation for the following problem

$$\sum_{t=0}^{\infty} \delta^t U^*(p_t) \longrightarrow \max \quad . \tag{20}$$

The discounted sum in (20) is maximized over all the trajectories applied to the conjugate technology from some initial point p_0.

Trajectory p_t given by (19) is optimal in the sense of criterion (20). Therefore

$$I^*(p_0) = \sum_{t=0}^{\infty} \delta^t U^*(p_t) \quad . \tag{21}$$

The following theorem describes the reciprocity of trajectories of direct and dual problems.

Theorem 4 (6). *Let $\zeta = \{x_t, c_t\}$ be an optimal trajectory (3.7) of equation (3.5) and let price vector p_0 be I-reciprocal with initial point x_0. Then, trajectory (19) is reciprocal with ζ in the following sense: for all t=0,1,... pairs (x_t, p_t) are I-reciprocal, pairs (c_t, p_t) are U-reciprocal and the following condition holds*

$$p_{t+1}x_{t+1} = p_t(x_t - c_t) \ . \tag{22}$$

By iterating (22) we get the following equality:

$$p_T x_T = p_0 x_0 - \sum_{t=0}^{T-1} p_t c_t \ . \tag{23}$$

This is a financial balance: the total value of products at final time T is equal to the total value of products at initial time 0 minus the total value of consumption during T periods. Taking into account the fact that $p_T x_T \approx \gamma^T \longrightarrow 0$ when $T \to \infty$, and taking a limit in (23), we get

$$p_0 x_0 = \sum_{t=0}^{\infty} p_t c_t \quad . \tag{24}$$

4.6 Strong turnpike hypothesis

Because of reciprocity of optimal trajectories, the turnpike hypothesis, as formulated in section 3.3, holds iff a similar turnpike hypothesis also holds for the conjugate problem. Trajectories (19) are created by transition map (18), that is, they are generated by generator function I^*. Because I^* is of a special kind (17), we will call the turnpike hypothesis for the transition map (18) the *weak* hypothesis.

Now we will formulate the *strong* version of the turnpike hypothesis, when the generator function is arbitrary in $\hat{W}$, not fixed to be of special form (17).

The strong turnpike hypothesis [4]. For any regular Gale technology ω and for any generator function $\Phi \in \hat{W}$, there is some ray (turnpike) such that all the trajectories of transition map

$$x \longrightarrow Y(x) := \operatorname*{Argmax}_{y \in \omega(x)} \Phi(y) \tag{25}$$

converge to it (in *angular* sense).[5]

This paper rejects not only the strong turnpike hypothesis but also the weak one (see section 5).

Note 6. *The strong hypothesis is formulated in space X but is applied to transition map (18) in the conjugate space X^*.*

Example 5. *(continuation of examples 1-4, see sections 2.1, 2.3, 3.3, 4.1). In the case of rectangular technology (11), independently of function I^*, transition map (18) is $Q(p) = pA$. This means that in (19) $p_t = p_0 A^t$. Further, from the Perron-Frobenius theorem it follows that*

$$(\alpha A)^t = (1/r(A) * A)^t \longrightarrow h\pi \qquad \textit{when } t \to \infty \quad ;$$

[5] $x \to y$ in angular sense means that $\cos(x, y) \to 1$.

(here, the rules of vector-matrix multiplication are used, hence $h\pi$ is a matrix) therefore $\alpha^t p_t \longrightarrow p_0(h\pi) = (p_0 h)\pi$. Thus, for large t trajectory $\{p_t\}$ moves along ray $R_+\pi$ at rate α^{-1}. Consequently, the strong turnpike hypothesis in the Rawls model with Leontief technology holds for any utility function.[6]

5 Counter-examples to the turnpike hypothesis

This section follows our work [9]. Our goal is to construct counter-examples to the strong and weak turnpike hypotheses. Note that for rejection of the strong hypothesis the weak counter-example is needed, and vice versa.

Counter-examples are constructed in two dimensional space $X = R^2_+$ for generalized Leontief technology

$$\omega(x) = \{y \in X| \ \ Ay \leq b|_{b=Bx}\} \ \ x \in X, \tag{1}$$

where A and B are nonnegative quadratic matrices (2*2). Matrix A is the same for all the examples, namely

$$A = \frac{1}{6}\begin{pmatrix} 16 & 1 \\ 4 & 16 \end{pmatrix}, \tag{2}$$

while B will be specific for each example.

Let $\Phi \in \hat{W}$ be the following Cobb-Douglas function

$$\Phi(x) = \frac{1}{2}x_1^{\frac{2}{3}}x_2^{\frac{1}{3}}, \ \ x = (x_1, x_2) \in X, \tag{3}$$

which is the same for all examples. The conjugate function to Φ is

$$\Phi^*(p) = \nu p_1^{\frac{2}{3}} * p_2^{\frac{1}{3}}, \ \ p = (p_1, p_2) \in X^*, \ \ \nu := 3 * 2^{\frac{1}{3}}. \tag{4}$$

[6] *Independently this result was pointed out in [6,7] and by Kaganovich in [16], see section 6.3 below.*

Denote the rows of matrix (2) by $a_i \in X^*$ $i = 1, 2$. They are normalized so that

$$\Phi^*(a_1) = \Phi^*(a_2) = 1. \tag{5}$$

Because the convergence in an angular sense is examined, every vector $x \in X$ can be characterized by an *angular coefficient* of the ray it belongs to

$$k(x) := x_1/x_2 \qquad x = (x_1, x_2) \in R^2_+.$$

For any scalar $s \geq 0$ we denote by $\bar{s}$ vector $(s, 1) \in X$. Vectors x and $\bar{s}$ are *incident* (i.e. they belong to a common ray) iff $s = k(x)$.

From (4.2) and (5) it follows that

$$1 = \frac{1}{\Phi^*(a_i)} = \max_{x \in X} \frac{\Phi(x)}{a_i x} = \max_{s \geq 0} \frac{\Phi(\bar{s})}{a_i \bar{s}} \quad i = 1, 2 \quad . \tag{6}$$

The maximum in the right side of (6) is reached at the rays that are incident with grad $\Phi^*(a_i)$ $\; i = 1, 2$, see (4.4). It is convenient to take the following vectors as representatives for these rays:

$$z^1 = (1, 8) \ , \ z^2 = (8, 1). \tag{7}$$

Note α_i and grad $\Phi(z^i)$ are incident vectors (see 4.6). These preparatory data are shown in Figure 1.

5.1 Transition map

Transition map (4.25) in the case of technology (1) is

$$x \longrightarrow y \in Y(b) \mid_{b=Bx} \quad Y(b) := \operatorname*{Argmax}_{Ay \leq b} \Phi(y). \tag{8}$$

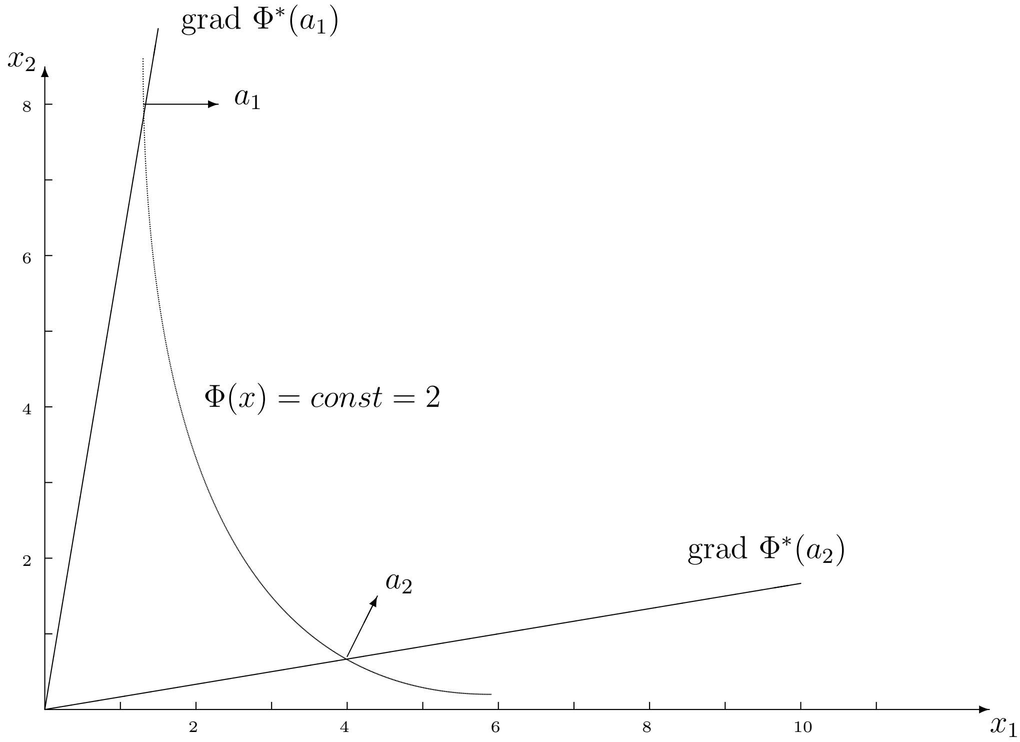

Figure 1: Illustration for the preparatory data

$Y(b)$ can be rewritten as

$$Y(b) = \underset{\max\limits_{i=1,2} \frac{a_i y}{b_i} \leq 1}{\text{Argmax}} \Phi(y) = \underset{y \in X}{\text{Argmax}} \frac{\Phi(y)}{\max\limits_{i=1,2} \frac{a_i y}{b_i}} = \underset{y \in X}{\text{Argmax}} \min \left[b_1 \frac{\Phi(y)}{a_1 y}, b_2 \frac{\Phi(y)}{a_2 y} \right]. \tag{9}$$

It is clear that the right side of (9) depends only on the angular coefficient of vector b, which we denote by $\beta = k(b)$. Then,

$$\gamma := k(y) = k(Y(\bar{\beta})) = \underset{s \geq 0}{\text{Argmax}} \min \left[\beta \frac{\Phi(\bar{s})}{a_1 \bar{s}}, \frac{\Phi(\bar{s})}{a_2 \bar{s}} \right] =: \kappa(\beta). \tag{10}$$

The function $\kappa(\beta)$ defined above sets an angular transition rule $\beta \to \gamma$ for the Leontief technology transition map $b \longrightarrow y = Y(b)$. Calculation of $\kappa(\beta)$ with the help of preparatory data (6) and (7) gives rise to the following result:

$$\kappa(\beta) = \begin{cases} 1/8 & 0 < \beta < 2/11, \\ \dfrac{16\beta - 1}{16 - 4\beta} & 2/11 \leq \beta \leq 43/16, \\ 8 & 43/16 < \beta. \end{cases} \tag{11}$$

Constants 1/8 and 8 are, as a matter of fact, the values of $k(z^i) = k(\mathrm{Grad}\Phi^*(a_i)),\ \ i = 1, 2$, see (7). Function κ is the key point in our analysis; its graph is shown on the fragment a) of Figure 2. The angular transition rule corresponding to transition map $x \to y$ is as follows

$$k(x) =: r \to \eta(r) := \kappa(\beta) \mid_{\beta=k(B\bar{r})} = \gamma = k(y), \tag{12}$$

see (8) and (10).

The strong turnpike hypothesis is equivalent to two properties of transition rule (12):

1) the fixed point of function $\eta(\cdot)$ is unique, that is the root r^* of equation

$$\eta(r) = r \tag{13}$$

exists and is unique;

2) all the trajectories, i.e. iterations of function $\eta(\cdot)$ from any initial point, converge to r^*.

Below, by choosing dependence $\beta = k(B\bar{r})$ (i.e. $b = Bx$) we construct different counter-examples to these properties.

5.2 Weak counter-examples to the strong turnpike hypothesis

Let us consider two simple examples, illustrated in fragments a) and b) of Figure 2.

Counter-example 1 (nonuniqueness of the fixed point). Let B be a unit matrix i.e. let technology (1) be a usual Leontief technology (2.7), then $b = x$ and $\eta(r) = \kappa(r)$.

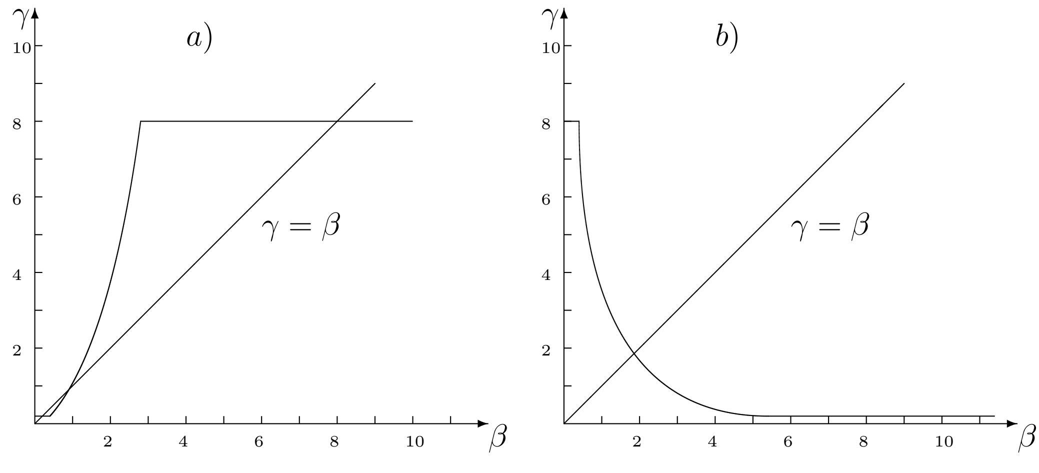

Figure 2: The counter-examples for the strong turnpike hypothesis

In this case equation (13) has three roots $r^1 = 1/8$, $r^2 = 1/2$, $r^3 = 8$, see fragment a) of Figure 2. Since at the point $r = r^2$ the graph of function η intersects diagonal $\eta = r$ from below, derivative $\eta'(r^2)$ is greater than one. This means that root $r = r^2$ is unstable and trajectories $r_{t+1} := \eta(r_t)$ do not converge to it. When $r_0 > r^2$ trajectories converge to root r^3; on the other hand, when $r_0 < r^2$ trajectories converge to root r^1. Note, that because function η in the neighborhoods of r^1, r^2 is constant, every trajectory converges with a finite number of steps.

Counter-example 2 (fixed point is unique, but nonstable). This example is based on the previous example and is just slightly modified. Now let:

$$B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{i.e.} \quad b = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

If B is thought of as an output matrix, it can be interpreted as we have just exchanged output rows; therefore this model is equivalent to the Leontief technology model, with matrix A obtained from (2) by a transposition of rows.

In this case $k(b) = 1/k(x)$ and consequently $\eta(r) = \kappa(1/r)$, see fragment b) of Figure 2. Function η is decreasing, therefore equation (13) has the unique root $r = r^* \approx 1.098$. Since the absolute value of $\eta'(r^*) \approx -1.368$ is greater than one, root

r^* is unstable. Trajectories move from the neighborhood of this root to a stable cyclic pair - roots of equation $\eta^2(r) := \eta(\eta(r)) = r$. It is easy to see that roots r^1 and r^3 from counter-example 1 constitute this cyclic pair. This follows from the fact that $r^1 r^3 = 1$.

5.3 Strong counter-examples to the weak turnpike hypothesis

To construct a strong counter-example we need to find some suitable Rawls model, that is a triplet $\{\omega, U, \delta\}$, see section 3.1. Let ω be a rectangular technology of type (4.11)

$$\omega(x) = \{y \in X \mid y \leq Ax\} \quad x \in X \quad , \tag{14}$$

where A is defined in (2). Then the conjugate technology is Leontief and coincides with the technology described in counter-example 1 of section 5.2 (in dual space!)

$$\omega^*(p) = \{q \in X^* \mid qA \leq p\} \quad p \in X^* \quad .$$

Further, we choose U and δ so that the corresponding normative utility function I coincides with function (4) (of direct argument $x \in X$ instead of $p \in X^*$!). In this case the conjugate function I^* coincides with function (3) (with the dual argument $p \in X^*$). Moreover, trajectories of the conjugate model, generated by (4.19), are exactly trajectories of counter-example 1 (in the dual space).

To reach this goal, function U^* and scalar δ have to satisfy equation (4.14), that is we choose

$$U^* := I^* - \delta \Gamma^* I^* \big|_{I^*(p) = \frac{1}{2} p_1^{2/3} p_2^{1/3}} \qquad \Gamma^* I^*(p) := \max_{qA \leq p} I^*(q). \tag{15}$$

Function U^* from (15) has to be monotonic and concave. Analytical calculations show that this is the case for $\delta < 0.027$.

If we transpose the rows of matrix A, then we get a similar counter-example on the basis of the second weak example in section 5.2.

5.4 Conditions under which the strong hypothesis is valid (two-dimensional case)

What conditions have to be imposed on technology ω to ensure that the turnpike hypothesis is valid? The general answer to this question has not been derived yet. However, in the two-dimensional case, this problem is simplified by the fact, that the $x \to y$ transition can be substituted, as was carried out in section 5.1, by the angular transition $r \to \gamma = \eta(r)$ making it a one-dimensional problem.

The convergence conditions could be expressed in terms of *supporting* function H of technology ω

$$H : \qquad H(x,p) := \max_{y \in \omega(x)} py \qquad x \in X, \;\; p \in X^*. \tag{16}$$

For fixed p function $H(\cdot, p)$ is an element of $\hat{W}$, while for fixed x function $H(x, \cdot)$ is an element of $\check{W}$. We assume that the technological cone (2.1) is strictly convex and function H is differentiable. Variables (x_1, x_2, p_1, p_2) are put in order as a united list $d = (d_i,\ i = 1, 2, 3, 4)$, $d \in D := X * X^*$. We use a subscript index as a sign of taking a derivative from the function H with respect to the corresponding variable. The following two lemmas facilitate the analysis of this case.

Lemma 6. *Trajectories of transition function $r \to \eta(r)$ converge (in the broad sense, including infinity) when the following symmetric conditions are valid*

a) if $s > r$ and $u \in [r, s]$, then $\eta(u) > r$;

b) if $s < r$ and $u \in [s, r]$, then $\eta(u) < r$;

where $s := \eta(r)$.

Proof. The conditions of this lemma state that, if at some point r the trajectory moves to the right (to the left), then at every later stage the trajectory is always on the right (on the left) of r. It is clear that such trajectories either go to infinity or converge (cycles are impossible). □

Lemma 7. *1) If everywhere in four-dimensional space D the conditions*

$$H_{14} > 0, \qquad H_{23} > 0 \tag{17}$$

hold, then the fixed point of the transition function η is unique (convergence is not clarified).

2) If everywhere in space D the conditions

$$H_{13} > 0, \qquad H_{24} > 0 \tag{18}$$

hold, then trajectories of function η converge (cycles are impossible, the fixed point can be nonunique).

Proof of this lemma is given in the Appendix.

Note 7. *From the properties of linear homogeneity and the monotonicity of function H with respect to x and p it follows that, at every point of space D, at least one of the pairs of conditions (17) and (18) holds.*

Corollary 5. *If everywhere in D both conditions (17) and (18) hold, the strong turnpike hypothesis is valid.*

Note 8. *In the case of the strong counter-examples with technology (14), supporting function $H(x, p) = pAx$. It follows that*

$$H_{13} = a_{11}, \;\; H_{23} = a_{12}, \;\; H_{14} = a_{21}, \;\; H_{24} = a_{22}.$$

As $A > 0$ conditions (17) and (18) hold in this case. However, there is no convergence because not all assumptions are satisfied. In particular, the technological cone is not strictly convex in this example.

6 Discussion

Let us outline some questions that can be discussed in the context of the results obtained.

6.1 Comparison with the average criterion

As it was mentioned in the introduction, for the average criterion the 'Consumption Turnpike Theorem' holds. On the other hand, in the model with the Rawls criterion this property does not hold. So, a reasonable question is: what is the crucial difference between these models? That is, what peculiarity makes the Rawls model reject the strong turnpike hypothesis?

A possible answer to this question is as follows. The average criterion under an infinite time horizon maximizes the asymptotic level of consumption; therefore, any finite part of a trajectory has zero weight. In this case it is obvious that the limit of a trajectory does not depend on the initial conditions. On the contrary, for the model with the Rawls criterion all time moments have the same consumption level, and therefore each of them is essential.

6.2 About strong counter-examples

Examples from section 5.3 show that in the Rawls model it might be the case that a number of attracting rays (attractors) exist. In the strong counter-example based on example 1 from section 5.2, there are two attractors. Depending on the initial conditions optimal trajectories converge to one of these rays. On the other hand, in the strong counter-example based on example 2 from section 5.2, optimal trajectories are attracted to a cyclical pair of rays. This means that each optimal trajectory alternates the structure of optimal values of production, consumption, etc.

6.3 Comparison with and comment on Kaganovich's results

To our knowledge, the closest publication in terms of subject matter and results is paper [16]. That paper considers a particular case of the Rawls model described here in which the technology ω is Leontief; one of the results of [16] was mentioned in footnote 6.

The main result of Kaganovich (2000) is that an adoptive rolling plan converges to the turnpike in the case of Leontief technology. To understand his analysis, let us formulate the notion of a rolling trajectory using our terminology. The rolling value function is[7]

$$R(x) := \max_{c,d} \min[U(c), U(d)] \quad x \in X \tag{19}$$

$$\text{s.t. } y \in \omega(x - c), \quad x \in \omega(y - d). \tag{20}$$

The rolling transition map $x \to y = Y(x)$ is determined by an optimal value of y in (20). A trajectory generated by the rolling transition map is denoted z_t.

Let us state two main properties of a rolling trajectory. Following the logic of [16], first, one can see that z_t is feasible as a solution for z_{t+2} in $R(z_{t+1})$. This means $R(z_{t+1}) \geq R(z_t)$, i.e. $R(z_t)$ is increasing with t. Second, z_t is feasible as a solution for x_t in $I(x_t)$, i.e. $R(x) \leq I(x) \; \forall \; x \in X$.

Kaganovich considered the Leontief technology case. In this case there is a turnpike in the model with the Rawls criterion (see section 3.3, example 1). The question Kaganovich tried to answer was whether a rolling trajectory converges to this turnpike. The main result of [16] is that the answer is positive. The proof is based on the claim that $I(z_0) \leq I(z_1)$, i.e. sequence $I(z_t)$ is decreasing. Unfortunately, there is no formal proof for this inequality. Moreover, our economic intuition suggests the opposite property. Specifically, the Rawls trajectory x_t has a constant level of consumption in periods $1, 2 \ldots, \infty$. Now, if we decrease the Rawls level of consumption to the rolling level of consumption in the first period ($R(z_0) < I(z_0) = I(x_0)$), then the Rawls level of consumption ($I(z_1)$) in state z_1 will be higher than the initial level

[7]Following Kaganovich (2000), we consider a 2-period rolling plan and assume $\delta = 1$.

$(I(z_0))$. Furthermore, as a specific example one can show that for Leontief technology with utility function $U = \Lambda_b$, sequence $I(z_t)$ is increasing.

The assumption that sequence $I(z_t)$ is decreasing is crucial for the proof. In that case, increasing sequence $R(z_t)$ is bounded and consequently must converge. This is the main result of (16).

Once again, the result was proved only for Leontief technology and only under the assumption that the sequence of $R(z_t)$ is bounded. In a general case, the following sequence $x, y, z, \alpha x, \alpha y \ldots$ where $\alpha > 1$, cannot be rejected by any results or theorems we are familiar with at this stage. Thus, whether an adoptive rolling-plan mechanism converges to the turnpike is an open question.

Finally, as an area of future research, it might be interesting to classify Rawls models by addressing the following questions:

1) is there a turnpike in a model?

2) if 'Yes', does the rolling plan converge to it?

The first question was partly discussed in section 5.4, however, there is considerable scope to explore this issue further.

A Proof of Lemma 4

From (4.7) and (4.2) it follows that

$$D(S) = \{x_1 + x_2 | \ \Phi_1(x_1) \geq 1 \ , \ \Phi_2(x_2) \geq 1\} = D(\Phi_1) + D(\Phi_2)$$

(the sum of sets in Minkowski sense). Next, using definition (4.1), one can derive (4.8,a). Equality (4.8,b) is a variant of the general result of the Moreau-Rockafellar theorem (see [21]), applying to (4.8,a). □

B Proof of Lemma 7

1) From the theory of conjugate functions (see sections 4.1 and 4.2) it follows that if x and y are connected by transition map (4.25), then there is a vector p such that

$$a)\ p \sim \mathrm{Grad}\Phi(y) \quad b)\ y \sim \mathrm{Grad}_p H(x, p). \tag{1}$$

Condition $a)$ describes the relationship between angular coefficient $\Theta := k(p)$ and $\gamma = k(y)$, that is

$$\Theta = k(p) =: \mu(\gamma). \tag{2}$$

Concavity of function Φ implies monotonicity of function μ, i.e. $\mu' \le 0$. From the fact that in the fixed point $x \sim y$ and by virtue of (1b)

$$x \sim \mathrm{Grad}_p H(x, p) \Longrightarrow r = k(x) = k(\mathrm{Grad}_p H(x, p)) = H_3/H_4.$$

Rewriting this equality as

$$H_3/k(x) = H_4 \tag{3}$$

and using homogeneity properties of functions H_3 and H_4 we get from (3)

$$H_3(1, 1/r, \Theta, 1) = 1/r H_3(r, 1, \Theta, 1) = H_4(r, 1, \Theta, 1) = H_4(r, 1, 1, 1/\Theta). \tag{4}$$

Function H is convex with respect to p. This means that under the condition that r is constant, the right side is a decreasing in Θ function, while the left side is an increasing in Θ function. As was mentioned, Θ decreases in $k(y)$ and consequently in $r = k(x)$ in fixed points. Therefore, under conditions (5.16) the left side of equation (3) decreases in r, while the right side increases in r. Thus, there is only a unique solution for equation (3).

2) Let us prove that under (5.17), conditions a) and b) of Lemma 7 are valid.

Imagine that condition a) of this lemma does not hold, i.e.

$$v := \eta(u) < r < u < s = \eta(r).$$

Note that, similarly to equation (4), the following equation for pair (r, s) is valid

$$s = H_3(r, 1, \mu(s), 1)/H_4(r, 1, 1, 1/\mu(s)) = \frac{1}{r} * H_3(r, 1, \mu(s), 1)/H_4(1, 1/r, 1, 1/\mu(s)),$$

or

$$H_3(r, 1, \mu(s), 1) = rs * H_4(1, 1/r, 1, 1/\mu(s)).$$

Dividing this equation by a similar equation for pair (u, v), we get

$$H_3(r, 1, \mu(s), 1)/H_3(u, 1, \mu(v), 1) = \frac{rs}{uv} * H_4(1, 1/r, 1, 1/\mu(s))/H_4(1, 1/u, 1, 1/\mu(v)).$$

Under conditions (5.17) the expression on the left side is less than 1, while the expression on the right side is greater than 1. This contradiction proves condition a) of Lemma 6. Condition b) can be proved in a similar way. □

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