Bundling in auctions with complementary goods*

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Abstract

We examine when a revenue-maximizing auctioneer prefers to auction a homogenous product in one bundle (a single-object auction) as compared with selling the item in two or more shares. When the items are super-additive (complementary) the auctioneer always prefers a single-object auction. When the product is sub-additive (supplementary), the auctioneer is more likely to choose a share auction when there are a large number of potential bidders. When there are a small number of bidders the auctioneer will tend to prefer a single-object auction. Further, we design a graphical approach for analyzing the optimal number of shares when there is a large number of bidders.

Key words: bundling, auctions, revenue maximization.

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1 Introduction

A vendor will try to choose the most advantageous way to sell their product. This could involve many aspects, such as marketing and the timing of the sale. If the product is divisible, a practical question a seller often faces is how many pieces (or bundles) should be used to sell the item; that is, should a seller use separate auctions or sell the products as one item? For example, the owner of a large area of land must decide how many lots to sell and what their size should be. One alternative, of course, is to sell the whole area as one lot, but often this is not what is observed. Similarly, should the Federal Communications Commission include two geographic regions under a single radio-spectrum licence when it is auctioned (Chakraborty 2006)? In this paper we consider how many bundles a seller will want to auction off a homogenous (divisible) good when she takes the auction process as given.

Analyzing an auction with heterogenous goods, Chakraborty (1999) showed that for a small number of bidders one auction would generate higher expected revenue; conversely, with a large number of bidders expected revenue is maximized using a share auction. The intuition is that with a small number of bidders, bundling stimulates competition for the items. In contrast to Chakraborty (1999), we study a model in which the product is homogenous, so that a bidder's valuation of separate items that make up a bundle are perfectly correlated. Furthermore, in Chakraborty (1999) bidders have additive values for their product valuation (the value of two items sold in one lot is the sum of their values if they were sold separately). Specifically, he does not allow for super-additivity or sub-additivity with respect to the valuations of the bidders. We explicitly do this; in doing so we directly address a research question highlighted by Chakraborty (1999, p. 730).

Wambach (2002) derived a formula for calculating the revenue generated in a share auction. The formula derived implies that revenue would be maximized by selling only one licence to create a monopoly. This follows because Wambach (2002) assumed that a buyer's valuation is directly proportional to the share received. This is a strong assumption - and one that turns out to be crucial. Allowing for a more general relationship as to how complementary the components of the bundle are and, in turn, how that affects the seller's choice of the revenue-maximizing auction, is one of the main contributions of the paper.

As in Chakraborty (1999), in our model the optimal choice regarding whether to adopt a share auction depends on the number of bidders when the products are sub-additive, or supplements using the terminology of Chakraborty (1999, p. 730). If there are a large number of potential bidders in the market, the auctioneer may find it preferable to sell the object using a share auction. If, on the other hand, there are only a small number of bidders a revenue-maximizing auctioneer will choose to sell the object as a single unit. The intuition for this result is as follows. First, as in a second-price auction, the winning bidder is expected to pay the valuation of the next highest bidder - for example, the price for the first share is the second-highest bidder's bid (their valuation) and the price for the second share is the third-highest bidder's bid (their valuation). This suggests that dividing the good into shares has the tendency to reduce revenue. Second, a bidder's willingness to pay for an additional share can depend on how much of the good they have. If, for instance, a bidder's utility is concave in their share holding, she will have a declining willingness to bid for any additional share.[1] This effect suggests that, in this case, using a share auction will increase the revenue generated. Which of these influences dominates depends on the number of potential bidders. If the number of bidders is large, the expected difference between the second-highest valuation and the bidder with the third-highest valuation is expected to be small, as should be the difference between the third- and fourth-highest valuation bidders, and so on. As a result, the expected loss from using

[1] It is worth noting that a bidder may have an increasing willingness to bid for any additional share, strengthening the auctioneer's incentive to run a single-unit auction. These specific conditions are outlined below.

a share auction is decreasing in the number of bidders. Thus, when the number of market participants is large, the second effect is likely to dominate the first effect; in this case a share auction generates higher expected revenue. On the other hand, when the number of bidders is small, the expected difference between the second and third highest bidders' valuations is (relatively) large, generating a larger expected loss from employing a share auction. As a consequence, with a small number of market bidders the first effect is more likely to dominate the second, leading a revenue-maximizing auctioneer to adopt a single-unit auction.

For some goods, of course, a bidder's valuation for the good will be such that additional portions are super-additive, or complementary using the terminology of Chakraborty (1999). In this case the marginal valuation of additional shares are increasing in the quantity of the good already held. If the good is strictly complementary, the seller will always maximize revenue by selling the item in one single bundle.

Finally, some good could display characteristics of being complementary and supplementary at different bundle sizes. We develop a graphical approach to analyze the optimal number of shares and their sizes when there are many potential bidders (this simplifies the analysis as with a large number of bidders the expected difference between the highest- and second-highest bid, and so on, becomes arbitrarily small). In this case, the optimal share size should be set where the slope of the valuation function for the quantity of the good held is maximized.

It is worth noting several assumptions that are made in this literature. First, as in Chakraborty (1999 and 2006) and Wambach (2002) we consider the bundling choice of the seller when she takes the auction process as given. Our objective here is not to develop an optimal auction;[2] rather, our task is to examine the seller's decision concerning the bundling of the object when she has no control over the auction

[2]Jehiel et al (2006) showed that making the bundling decision before the bids are submitted is not optimal.

platform. For example, a public authority could be required by law or government policy to follow a set auction or procurement process, but they do have control over the number of shares it divides the total into. Similarly, a seller can choose how to divide up an item for sale on eBay, but then must adhere to the rules of the platform.

A second important assumption is that each buyer is only allowed to purchase one share at most. As evidenced by the sale of spectrum licences in several European countries, many auctions of divisible items split the total into parcels or shares and require that each bidder may not purchase more than one share. For instance, in order to generate product market competition the auction of 3G telecom licences in the United Kingdom involved four licences (see Binmore and Klemperer 2002). Similarly, in a reverse auction, a potential advantage of using several suppliers is spreading the risk of not receiving the input due to the closure of a supplier. Alternatively, in a procurement process, a firm may choose to use several sources for an input in order to avoid post-contractual opportunism from a sole supplier.

2 The model

The auctioneer wants to sell k shares of some object, $x_1, x_2, ..., x_k$ where $\sum_{i=1}^{k} x_i = 1$ and $x_i \geq x_j$ for $i < j$. There are n risk-neutral bidders with $n > k$. Each bidder l has a valuation of $v_l f(x_i)$. v_l is a privately known value that is drawn for all of the bidders from the same distribution $G(v)$ on $[\underline{v}, \overline{v}]$. $f(x_i)$ is the common component to all bidders of the valuation of share x_i, which is common knowledge. From hereon, we refer to v_l as the private component of bidder l's valuation and $f(x_i)$ as the common component. Note, Wambach (2002) assumed $f(x_i) = x_i$. Here we only assume that $f(\cdot)$ is twice differentiable, non-negative and non-decreasing; that is, $f(x_i) \geq 0$ and $f'(x_i) \geq 0$. In addition, for the purpose of normalization we assume that $f(1) = 1$.

First, we derive the expected revenue for the auctioneer in every efficient auction. An efficient auction is defined as an auction in which the bidder with the highest valuation ($v_{(1)}$) receives the largest share, the bidder with the second-highest valuation ($v_{(2)}$) receives the second largest share, and so on.

Proposition 1. *Every efficient auction leads to the following expected revenue R for the auctioneer:*

$$R = \sum_{i=1}^{k-1} i(f(x_i) - f(x_{(i+1)}))E[v_{i+1}] + kf(x_k)E[v_{(k+1)}] \tag{1}$$

where $v_{(i)}$ is the i-th highest valuation and $E[\cdot]$ represents the expectation.

This is a generalization of Proposition 1 in Wambach (2002), so the proof is omitted. The intuition for the result can be thought as a sequence of auctions. The first auction allocates a share of x_k to the k highest bidders using an ascending $(k+1)$-price auction: these k bidders pay the $(k+1)$-highest bid, yielding the second term on the right hand side of the equation above. Next, in the second auction the highest $k-1$ bidders are allocated an additional share of $(x_{k-1} - x_k)$ for the k-th highest bid; this procedure continues until the highest bidder is allocated an additional share $(x_1 - x_2)$ for the second-highest bid: this yields the first term in the equation determining R above. This is an efficient auction mechanism as each player is better off bidding their true value for each share; an alternative strategy results in a lower payoff; for example not bidding for a certain share gives a zero return, rather than the possibility of a positive profit. Given the Revenue Equivalence Theorem, this result applies for all efficient mechanisms.

Now we can analyze how and when the auctioneer should split the item to maximize revenue.

2.1 Complementary goods; super-additive valuations

The following proposition identifies the sufficient conditions under which an auctioneer wishing to maximize expected revenue should sell the item as a whole.

Proposition 2. *If* $f(x) \leq x \; \forall \; x$ *then a profit-maximizing auctioneer chooses* $x_1 = 1$ *(single-object auction).*

Proof. As $v_{(i)}$ is decreasing with i, the following inequality holds:

$$R < \sum_{i=1}^{k-1} i(f(x_i) - f(x_{(i+1)}))E[v_{(2)}] + kf(x_k)E[v_{(2)}] = \sum_{i=1}^{k} f(x_i)E[v_{(2)}], \tag{2}$$

where R is the expected revenue, as in equation 1. The expected revenue from selling the whole item as a single unit is $E[v_{(2)}]f(1)$. Consequently, a sufficient condition for generating a smaller expected revenue from a share auction compared with a single-object auction is:

$$f(x_1) + f(x_2) + \ldots + f(x_k) \leq f(1) = 1 \; \forall \; k, x_1, x_2, \ldots, x_k. \tag{3}$$

This condition is satisfied because $f(x_i) \leq x_i$ for $i = 1, 2, \ldots, k$
and $\sum_{i=1}^{k} x_i = 1$. □

The intuition for the proposition is as follows. When we compare a share auction with a single-object auction there are two effects that determine which format generates more revenue. The first effect relates to the private valuation of a bidder. As $v_{(i)} > v_{(i+1)} \; \forall \; i$, this effect always works in favor of a single-object auction. The second effect relates to the common component of a bidder's valuation. The contribution from a change in x_i depends on how the marginal benefit of the common component of a bidder's valuation - that is, $f'(x_i)$ - changes. In general, the second effect can either reinforce the first effect or it could work in the opposite direction. The specified conditions in Proposition 2 ensure that the second effect (the common component) works in the same direction as the first effect (the private component). It is worth noting that the effect of the common-valuation component is zero in the model of Wambach (2002) - because of this selling the item as a whole generated higher expected revenue.

The second effect will always reinforce the first effect if $f(x) \leq x \; \forall \; x$. In particular, this holds when $f(x)$ is convex, provided $f(0) = 0$. This will occur, for example, when monopoly profits that will accrue to the winning bidder in a single-object auction are higher than total industry profits with an alternative market structure (like an oligopoly).[3] Increasing returns with respect to the share that a bidder successfully obtains will have a similar effect; revenue will be higher for the auctioneer if the item is sold as a single item.

2.2 Supplementary goods: sub-additive valuations

When will a share auction maximize revenue? One situation when an auctioneer wishing to maximize expected revenue may find it optimal to sell the item in shares is when $f(0) > 0$. In this rather perverse case, the auctioneer will have an incentive to split the object into a very large number of parts and receive an arbitrarily large profit.[4]

Let us consider a more general case when $f(x) \geq x \; \forall \; x$ and $f(x) \neq x$ for some x. Now the second effect works in favor of a share auction, i.e. in the opposite direction to the first effect outlined in section 2.1 above. This means that in the general case either of two effects can dominate and it is not possible to determine whether a single-object auction or selling the item in several shares will result in higher expected revenue for the auctioneer. However, what is possible to ascertain is that the relative magnitude of the first effect diminishes with the number of participants in the market, while the second effect is invariant to the number of participants. Proposition 3 summarizes above discussion by outlining situations when an auctioneer prefers a share auction

[3] Clearly, a bidder's willingness to pay depends on their expected profits if their bid is successful. A single-object auction corresponds, in this context, to creating a monopoly; a share auction corresponds to creating a market in which there is more than one competitor.

[4] Note, however, the number of shares k has to be less than number of participants in the market n. Consequently, even in this case it is still possible that a single-object auction is optimal.

to a single-unit auction and vice versa.

Proposition 3. *If $f(x) \geq x \; \forall \; x$ there is an n^* such that for $n \geq n^*$ a profit-maximizing auctioneer chooses $x_1 < 1$ (share auction).*

Proof. As $E[v_{(k)}]$ converges to $\overline{v} \quad \forall \; k$ when $n \rightarrow \infty$, the difference $E[v_{(k)}] - E[v_{(k-1)}]$ converges to 0 $\forall \; k$ when $n \rightarrow \infty$. Therefore, from equation 1, the expected revenue from a share auction $R \rightarrow \sum_{i=1}^{n} f(x_i)\overline{v}$ when $n \rightarrow \infty$. Given that $\sum_{i=1}^{n} f(x_1) > \sum_{i=1}^{k} x_i = 1$, in the limit when $n \rightarrow \infty$ the expected revenue from a share auction is higher than the expected revenue from a single-unit auction, i.e. $\sum_{i=1}^{n} f(x_1)E[v_{(2)}] > E[v_{(2)}]$. In other words, the expected revenue from a share auction converges to a value which is higher than the expected revenue from a single-unit auction. □

The two effects - the common component and the private component - will always work in opposite directions if $f(x) \geq x \; \forall \; x$. In particular, this holds when $f(x)$ is concave. In this case the combined effect from the common and private components suggests that the marginal benefit from an increase in the size of the share is falling; this implies that there is no rent dissipation from increasing the number of competitors in the market created by the auction. This could be the case when the item for sale - a range of the spectrum for example - can be used as an input into the production of goods or services in different markets (and there are limited synergies between these markets). In such a case, splitting the divisible item into shares can increase, rather than decrease revenue.

When $f(x) \geq x \; \forall \; x$ and $n = 1$, a single-unit auction is preferred. When n is large, a share-auction will maximize revenue. For intermediate numbers of potential bidders, the preferred auction structure is not as clear. To help explore this issue we assume the following regularity condition is satisfied:

The Regularity Condition 1. *For any $m \in N$ and $k \in N$ where $m < k$, $f_{mk}(n) = \frac{E[v_{(k)}(n)]}{E[v_{(m)}(n)]}$ is a monotonically increasing function in n.*

The Regularity Condition has the interpretation that the convergence of $E[v_{k-1}]$ to $\overline{v}$ is relatively quicker than the convergence of $E[v_k]$ to $\overline{v}$. This is because the value of v_{k-1} is always smaller than the value of v_k for any n, while they converge to the same value $\overline{v}$. Most distributions commonly used, such as a power distribution, satisfy this Regularity Condition.[5] The condition allows us to prove the following Proposition.

Proposition 4. *If the Regularity Condition holds and if $f(x) \geq x \ \forall \ x$, there is an n^* such that when $n < n^*$ a single-object auction generates higher expected revenue than a share auction. For $n > n^*$ a share auction generates higher expected revenue.*

Proof. Compare the expected revenue from a share auction with the expected revenue from a single-object auction, described by equation 1. Specifically,

$$\sum_{i=1}^{k-1} i(f(x_i) - f(x_{(i+1)}))E[v_{i+1}] + kf(x_k)E[v_{(k+1)}] \sim E[v_2]. \tag{4}$$

Dividing each side by $E[v_2]$ yields

$$\frac{\sum_{i=1}^{k-1} i(f(x_i) - f(x_{(i+1)}))E[v_{i+1}] + kf(x_k)E[v_{(k+1)}]}{E[v_2]} \sim 1. \tag{5}$$

The left-hand side of 5 is monotonically increasing in n. For small n the left-hand side of equation 5 is less than 1. For large n the left-hand side of 5 is greater than 1. Thus, there exists a unique point of equality where $n = n^*$. As the left-hand side and right-hand side give the expected revenues of a multi-unit auction and a single-unit auction respectively, when $n < n^*$ a single-object auction generates higher expected revenue than a share auction and for $n > n^*$ a share auction generates higher expected revenue. □

[5] In fact we could not find any distribution that does not satisfy this regulatory condition.

2.3 An approach to the general case

In essence, Proposition 2 refers to the case when $f(x) \leq x \ \forall \ x$ and Proposition 3 considers the case when $f(x) \geq x \ \forall \ x$. There could be cases where goods are neither complementary nor supplementary. To simplify analysis of these cases let us assume that n is relatively large so that the second effect is relatively small. In particular, we consider the function shown in Figure 1.

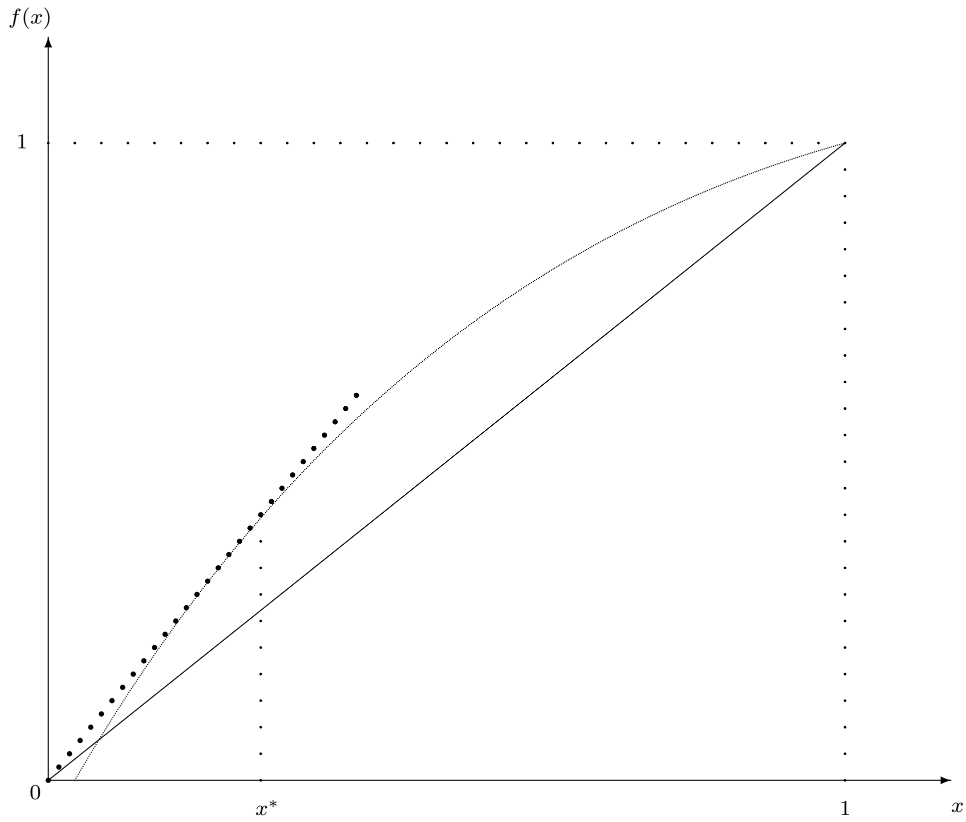

Figure 1: The revenue-maximizing number of shares

In Figure 1, a buyer's value of getting a small portion of the good is useless - for very small values of x, $f(x) = 0$. This practically means that there is a minimum size of each share. Further, we assume that $f(x)$ is any monotonically increasing function with diminishing marginal returns, i.e. $f'(x) \geq 0$ and $f''(x) \leq 0$.

The seller will now choose the optimal share size where the slope of $f(x)$ is the highest - this is where $x = x^*$. In this case the number of shares will be $1/x^*$. Of course, we are side stepping two potential complications in our graphical representation of the seller's problem. First, if the prescribed share is a significant portion of 1, there could be a discrete numbers problem. Second, if the prescribed share is small, it could require that the auctioneer sell many shares. When the number of shares to be sold is greater than the number of potential bidders ($k > n$) this technique will need to be revised.

Notably, this technique also applies when there are j different types of consumers, where each different type of consumer has a valuation function $f_1(x), f_2(x), \ldots, f_j(x)$. If there are many potential buyers in each group, the seller will maximize revenue considering the composite function $f = max[f_1, f_2, \ldots, f_j]$. As above, revenue will be maximized by setting shares so as to maximize the slope of the composite value function f.

3 Concluding comments and extensions

In this paper we have shown that a profit-maximizing auctioneer of a homogenous product may prefer a share auction to a single-unit auction. If bidder's valuations are super-additive (complementary) in the share of the object they obtain, as would be the case when a single-unit auction creates a monopoly for the winning bidder, revenue is maximized by selling the product as a whole. For bidder valuations that are sub-additive in the share received (supplementary), revenue could be maximized with a share auction. We show that in the case when the goods are supplementary, with a large number of potential bidders, it is more likely that a share auction will be preferred by the auctioneer. Conversely, with a small number of bidders a single-unit auction is more likely to maximize expected revenue. The analysis outlined here applies to a reverse auction; the seller may award contracts to several of the tenderers to supply a proportion of the total required. With certain supplier cost functions, it may be less costly for the purchaser to source the total quantity of the input required from more than one supplier.

We also develop a graphical approach to determine the revenue-maximizing number of shares when the good can be complementary or supplementary at different quantities.

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