1 Introduction

Economics and business school professors love to teach of first-mover advantages, but it is not always so. Tellis and Golder (1996) showed that in practice the second entrant into a new market often does better than the first firm that entered: for instance, the pioneer of color television, RCA, was leapfrogged by Sony and Matsushita (Mueller 1997).[1] The theoretical literature has examined a second-mover advantage in a variety of situations. For example, in a game with strategic complements, such as in a differentiated products Bertrand game, the leader is at a disadvantage to the second mover (Hamilton and Slutsky 1990 and Deneckere and Kovenock 1992).[2] A second mover could avoid contributing to a public good (Bliss and Nalebuff 1984) or avoid being held-up (Smirnov and Wait 2004).

In this paper we study endogenous leadership in a model of market entry in which the option to wait - that is, being the second mover - bestows an advantage on that player. To analyze this problem we adopt the war-of-attrition framework of Fudenberg and Tirole (1991, section 4.5) in which a player that can 'endure' waiting longer than their rival receives a higher payoff. Consider, for example, a game with two players who can each decide to invest (enter the market) or to wait in any period. Each player prefers to be a follower rather than a leader in the market, perhaps because they can free ride on the other party's investment. In addition, we allow a player's payoff to be different if it is the market leader - and sole mover in that period - as compared with the situation in which it invests at the same time as the other player.

We assume that each player has an option to wait - that is not enter the market - for a maximum of T periods; T can be thought of as a deadline. Once the deadline has been reached investment is no longer possible. A deadline could arise if, after a certain delay without entry, other (less technologically advanced) firms will be able to enter and capture the market, preventing profitable entry in the future by either of the duopolists considered here. Consequently, the imminent threat of entry by these new rivals can provide a deadline for profitable entry. Alternatively, the firms may know that the government will remove duopoly protection of a market if entry has not occurred by a particular time. Another possible example of a deadline is a special event, such as the Olympics; a firm could establish itself before the event, but the date of the event itself puts a very definite deadline on profitable entry. In the model we assume that there is perfect knowledge about the time remaining before the deadline. The time the players have before the deadline turns out to be an important factor in the model.[3] Once at least one player has moved

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[1] For an extensive discussion of second movers outperforming industry pioneers see Tellis and Golder (1996) and Mueller (1997).

[2] Gal-Or (1985) showed that a follower makes higher profits relative to the leader if the reaction functions are upward sloping.

[3] As pointed out by a referee, while we focus here on different possible deadlines for

at any stage, the other player will adopt their best response immediately (in the next period). The tradeoff for each firm is then to be the first mover with a short lag (avoiding costs of delay) or to be second, but potentially endure a long wait. This tradeoff will depend on the length of time before the deadline.

Despite the apparent simplicity of the model, some of the results of the game are striking. First, we show that there is a unique symmetric stationary equilibrium in the infinite-horizon game (with no deadline) in which each player mixes between investing and waiting in every period until entry has occurred. Next, we examine model when there is a deadline. If the deadline is short and the surplus from immediate entry (as a leader) is relatively small compared with the potential gain from free-riding, each player will have an incentive to wait. In this situation the game has the structure of a prisoners' dilemma, with both parties staying out of the market and only entering at the deadline. However, extending the time until the deadline reduces the payoff from waiting and makes immediate entry relatively more attractive. When the deadline is sufficiently remote, in the symmetric equilibrium each player adopts a mixed strategy entering the market immediately with a positive probability.

If the deadline is extended further the players could again have a dominant strategy to wait. The intuition for this result is as follows. The payoff to each player if they both wait with a T-period deadline is the payoff in the $(T-1)$ game, appropriately discounted. Assume in the $(T-1)$-deadline game the players adopt a mixed strategy. If this mixed-strategy payoff is sufficiently large, a player will prefer to wait with T periods remaining, in anticipation of this mixed strategy payoff. In this case, a player does not invest with a deadline of T periods (has a dominant strategy to wait) but invests with a positive probability in a $(T-1)$-deadline game (mixed strategy). Following this logic, the probability that a player invests can switch from positive to zero and back again as the deadline (T) changes. Furthermore, this switching as the deadline is altered can move periodically or non-periodically as the number of periods is extended. It is shown that the unusual dynamics - switching non-periodically from a mixed strategy to waiting and back as the deadline horizon is extended - can be chaotic.[4] As it turns out, the relative payoffs of the leader and follower compared with the simultaneous entry payoff play a crucial role in determining whether there are chaotic trajectories. In the limit when the discount factor goes to 1, chaotic behavior occurs when it is socially optimal for the players to enter the market simultaneously, rather than one after the other. As we discuss in section 5, this relationship coincides with the condition that the efficiency effect does not hold - in this context this means that the total payoff to two firms that simultaneously enter the market is larger than the profit of a monopolist that entered the market alone.[5] This suggests

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profitable entry, the same results can arise in the case when the deadline is fixed and the discount factor is changed.

[4] Medio (1992) provides a survey of applications of chaos to economics.

[5] See Tirole (1988, p. 393) for a definition of the efficiency effect.

that we could expect different trajectories in different industries: industries in which spillovers are particularly important are more likely to display chaotic trajectories, other things equal. Consequently, the main findings of the model may well be of interest if taken to the data.

2 A market-entry game

To explore the set-up of the model, consider the following scenario. Two firms, A and B, are contemplating entering an industry in which there are set-up costs in either developing the product or creating a new market. These costs are borne by the firm (or firms) that enter first and are sunk once they have been made. That is, if both firms enter simultaneously they will share the costs. If not, the costs will be incurred solely by the leader. In the case of sequential entry, the second-mover avoids incurring these set-up costs and can free ride on the first-mover's investment.

If entry has not already occurred, a party can either invest (I) or not invest and wait (W). If both parties choose I, each party enters the market, shares the industry set-up costs and receives a payoff of v_S. If a party invests and the other party opts for W, the first party receives a payoff of v_L. This can be thought of as a situation in which one firm enters the market alone, bears all the development costs but has a monopoly for that period. The party that did not enter has to wait until the next period. By construction this party will now opt to enter the market immediately once the sunk costs has been incurred by the other firm, and this second-moving party receives a payoff of v_F. By assumption $v_F > v_S > v_L$.

If both parties choose W, unless it was the deadline the game proceeds to the next period. In that case, the process outlined above is repeated, with all payoffs discounted by δ, where $1 > \delta > 0$.[6] In the game we denote t as the time period and T is the deadline or potential time horizon. Once the deadline T has been reached both parties receive x_1.

As noted above, we focus on a discrete-time model. As argued by Fudenberg and Tirole (1985) in their preemption game, discrete periods are appropriate for this market-entry game. Periods could be discrete because of technical or bureaucratic reasons; for example, the decision to enter a new market may need to be ratified by a board of directors that only meets periodically or due to the seasonality of product demand. Alternatively, Levin and Peck (2003) interpret a period as the length of time that must elapse between when a firm enters and when their rival can observe this move. Furthermore, the assumption that a firm entering the market at the same time as its rival earns a different payoff (v_S) to when it enters first (v_L) only makes sense with discrete-time periods.

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[6] A different approach of introducing some costs k from waiting each period does not change the results qualitatively.

An interpretation of the model

As an example to motivate the model, consider two firms that are both contemplating whether to release a new type of computer-game console. The new design is not patentable in that it can easily be replicated for all practical purposes by the follower. Further, consumers are initially unaware of the benefits that the new console has and how it will enhance the computer-game playing experience. Consequently, the firm or the firms that first introduces the console to the marketplace will need to promote it; this cost will be borne by the market leader, or if both firms enter simultaneously they will both contribute to these promotion costs. Let the costs associated with establishing the market be denoted by K. Clearly, a follower will avoid these promotion costs, but will be able to enter the market and free ride on the leader's investment, providing it with a second-mover advantage.[7] Given this, once one firm has entered the market the other will enter at the next opportunity.

We assume that once both firms have entered the market they act as infinitely-long lived duopolists. This could arise if there is sufficient demand only to sustain two firms in the market. Further, given there is room for two firms in this market, there could effectively be a deadline on profitable entry: if either firm delays entry too long a third firm will eventually generate the necessary know-how to enter the market and install itself as one of the incumbent duopolists. In this way, it is not inconsistent in the model that the firms receive an infinite stream of payoffs after entry, but possibly face a finite deadline for profitable entry. Finally, each firm can decide to enter or not in a given period. This discreteness could arise due to the requirements arising from the seasonality of retail trading. For example, there may be a window of opportunity to get the consoles into retail outlets before the Holiday shopping season (along with the accompanying promotion), that if missed will require a firm to wait until the next season (the next available opportunity to enter).

We are now in a position to compare the payoffs of a market leader, a follower and the payoff to a firm if there is simultaneous entry. Let π_m be the one period monopoly profit and let π_d denote the one-period duopoly profit for each firm if they are both in the market. In this case, if one firm enters the market alone it earns a return of $\pi_m - K + \frac{\pi_d \delta}{1-\delta}$; this is equal to v_L. In this case the market leader gets monopoly profits in the first period, but must bear the set-up costs K alone. After the market has been established, the other firm will enter in the next period, creating a duopoly. Alternatively, the return to a firm that is a follower will be $\frac{\pi_d \delta}{1-\delta}$, which will be equal to v_F. The assumption that $v_L < v_F$ will be satisfied provided that $\pi_m - K + \frac{\pi_d \delta}{1-\delta} < \frac{\pi_d \delta}{1-\delta}$, or that $\pi_m < K$. Finally, the return to a firm if there is simultaneous entry (v_S) is $\pi_d - \frac{K}{2} + \frac{\pi_d \delta}{1-\delta}$. In this way, as assumed in the model above, the payoffs depend only on the order

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[7] This example has similarities to the investment RCA made promoting color television, notably through its relationship with NBC, essentially creating a market that the followers could exploit (see Tellis and Golder 1996).

of entry (discounted for the appropriate entry delay). Further, the follower's payoff is higher than the payoff to each firm from simultaneous entry provided $\pi_d < \frac{K}{2}$. In turn, the simultaneous entry payoff is higher than the payoff to a firm that is a market leader provided $\pi_m - \frac{K}{2} < \pi_d$. This gives the order of payoffs assumed above; $v_F > v_S > v_L$.

3 Market entry with no deadline

Consider the game when there is no deadline, so that the potential horizon is infinite (that is, $T = \infty$). The extensive-form game is illustrated in Figure 1. Here, A can choose to invest immediately (I) or can choose to wait (W). At the same time B has the strategic options of investing (I) and not investing and waiting (W).

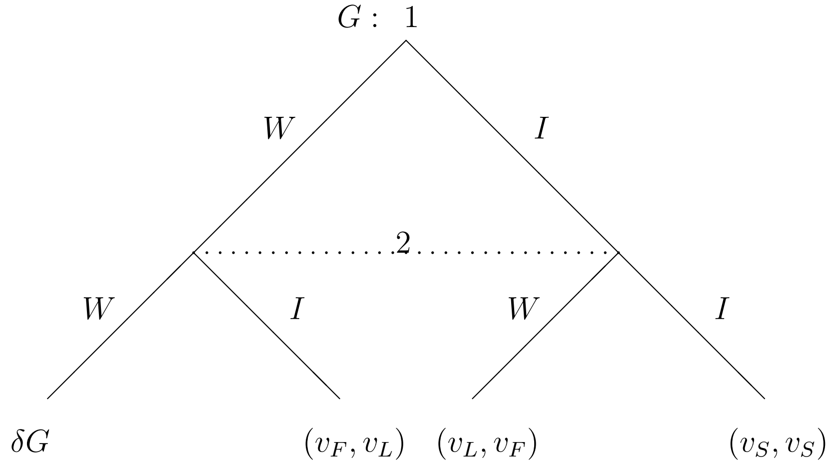

Figure 1: Extensive form for the relevant infinite-horizon game

There are three stationary subgame-perfect equilibria (SPE) in this game.[8] The first involves B investing immediately and A always following. A symmetrically equivalent equilibrium exists in which A invests immediately and player B waits.

The focus in this paper, however, is on the more interesting symmetric equilibria.[9] In the infinite-horizon model a stationary mixed-strategy equilibrium

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[8] If players adopt the same actions in every period (until entry occurs) we refer to this strategy as stationary. As discussed further below, there are an infinite number of subgame-perfect equilibria in the infinite-horizon game.

[9] As argued by Dixit and Shapiro (1986) in their entry game, asymmetric equilibria are arbitrary and unexplained. Farrell and Saloner (1988) claimed that asymmetric pure-strategy Nash equilibria give the outcome of the game without any modelling of the process or institutions that facilitate this coordination. Crawford and Haller (1990) argued that asymmetric pure-strategy equilibria are not convincing. Also see Farrell (1987) and Bolton

exists. In this equilibrium both parties invest with some positive probability. For example, player A invests immediately with probability α and player B invests immediately with probability β. The payoffs to each player are outlined in Figure 1. However, if both parties do not invest in the first period, which occurs with probability $(1-\alpha)(1-\beta)$, the players return to an identical situation, only one period in the future. A unique stationary symmetric Nash equilibrium always exists, as stated in Proposition 1.

Proposition 1. *In the infinite-horizon game a unique non-degenerate stationary symmetric Nash equilibrium exists.*

Proof. In the mixed strategy equilibrium, each player must be indifferent between investing and waiting. Furthermore, symmetry implies that $\alpha = \beta$, so for either player:

$$G = \beta v_S + (1-\beta)v_L = \beta v_F + (1-\beta)\delta G, \tag{1}$$

where G is defined as the value of the game. The resulting quadratic equation with respect to β has two solutions: the first is greater than 1, the second is between 0 and 1. The second is feasible as a solution to this problem, while the first is not, as β is a probability. As there is always one solution, a mixed strategy always exits. □

4 Market entry with a finite-horizon deadline

Now consider the game with a deadline. As specified in section 2, if no player invests before the deadline, each party gets a payoff of x_1. First, assume that the deadline is $T = 1$. The normal form for this game is illustrated in Figure 2.

		Player B	
		I	W
Player A	I	v_S, v_S	v_L, v_F
	W	v_F, v_L	x_1, x_1

Figure 2: The normal form of the $T = 1$ game

By assumption, $v_F > v_S > v_L$. Also, let $x_1 > v_L$.[10] Given these parameter values, the players face the prisoners' dilemma in which the Nash equilibrium

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and Farrell (1990).

[10] This assumption is made only to ease the construction of the model. It is not necessary and we will relax it later.

is (W, W), i.e. both players wait and receive payoff of x_1.[11]

Now extend this game so that there are two periods before the deadline $(T = 2)$. In both periods each player can either play I or W. In the first period, if both players adopt action I the payoffs are realized as previously mentioned, and equal v_S for each player. If one party invests and the other plays W the payoffs are given by (v_F, v_L) and (v_L, v_F) respectively, as illustrated in Figure 2 - again v_L and v_F. If both players adopt W, however, the game proceeds to the second and final period; that is, the game returns to the one-period game in Figure 2. Thus, the payoffs to each player if they both adopt W in the first period (at $t = 1$) of the two-period game is $x_{T=2} = \delta x_1$, where $x_{T=2}$ represents the equilibrium payoff to each player if they both select W with two-periods remaining until the deadline $(T = 2)$. For convenience, let $x_{T=2}$ be written as x_2. Provided $x_2 > v_L$, each player has an incentive to wait in the first period of this longer game, even though total surplus is diminished as a result - again the players are in a prisoners' dilemma.

The addition of a third potential investment period so that there are three periods until the deadline $(T = 3)$ will not alter the strategy of each player provided $x_3 = \delta x_2 = \delta^2 x_1 > v_L$. Further, for any T when $x_T > v_L$, the value for x_{T+1} can be calculated as

$$x_{T+1} = \delta x_T. \tag{2}$$

Note, the subscripts on x_T denote the number of periods until the deadline.

As the number of periods in which the players can opt to wait is increased, the payoff from waiting until the final deadline is reduced with each subsequent period. Eventually, $\delta^{T-1} x_1 < v_L$. If this is the case, the players will adopt a mixed strategy at $t = 1$ (in the initial period) of the T-deadline game.[12] As in the infinite-horizon game, we focus here on the symmetric equilibrium although non-symmetric equilibria also exist.[13]

Now consider extending the number of periods given the players opt for a mixed strategy with T periods until the deadline (that is $\delta^{T-1} x_1 < v_L$). When there are $T + 1$ periods remaining, if both players opt to wait, their expected payoff is the mixed-strategy payoff with T periods remaining, discounted by δ. As such, the payoff x_{T+1} from opting to wait with $T + 1$ periods remaining until the deadline when $\delta^{T-1} x_1 < v_L$ is $x_{T+1} = f(x_T)$ where[14]

$$f(x_T) = \frac{v_L v_F - v_S x_T}{v_L + v_F - v_S - x_T} \delta. \tag{3}$$

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[11] Note, if $x_1 \geq v_S$, the game is not a prisoners' dilemma, but the Nash equilibrium is still (W,W).

[12] In the mixed-strategy equilibrium the probability that each player will invest, p_T, is given by $p_T = \frac{v_L - x_T}{v_F - v_S + v_L - x_T}$.

[13] For example, an asymmetric equilibria could involve player A adopting the strategy always invest while player B adopts the strategy always wait.

[14] See the Appendix for the derivation.

Combining equations 2 and 3, one can derive a mapping where value of x_{T+1} is calculated with help of equation 2 when $x_T > v_L$ and with help of equation 3 when $x_T < v_L$, i.e.[15]

$$x_{T+1} = \begin{cases} \frac{v_L v_F - v_S x_T}{v_L + v_F - v_S - x_T}\delta, & \text{when} \quad x_T < v_L; \\ \delta x_T, & \text{when} \quad x_T > v_L. \end{cases} \tag{4}$$

Note, x_T is the expected payoff for a player when both wait in the initial period, play the symmetric equilibrium path in all subsequent periods until the deadline at T. Further, from equation 4 it is evident that, as long as x_1 is non-negative, the assumption that $x_1 > v_L$ can be relaxed. In particular, $x_1 = 0$ could mean that if the deadline is reached then the profitable opportunity disappears and both players get zero.

Now let us analyze mapping 4. If $x_T < v_L$, x_{T+1} is decreasing in x_T, as the function specified in equation 3 is decreasing with respect to x_T. Specifically,

$$f'(x_T) = -\frac{(v_F - v_S)(v_S - v_L)}{(v_L + v_F - v_S - x_T)^2}\delta < 0. \tag{5}$$

To further consider the relationship between the equilibrium strategies and the time horizon until the deadline T, refer to Figure 3. In the Figure, $f(x_T)$ is represented by the downward sloping line. If $x_T < v_L$ and $x_{T+1} < v_L$, in the initial period the players will play a mixed strategy with a time horizon of $T+1$ periods, as well as with T periods remaining. Define z_1 as the value that ensures that $v_L = \frac{v_L v_F - v_S z_1}{v_L + v_F - v_S - z_1}\delta$. In this case, provided $v_L > x_T > z_1$ the players will adopt a mixed strategy at $t = 1$ in the $(T+1)$-period game. Furthermore, provided that $z_1 < \delta v_L$, once the time horizon until the deadline is sufficiently long such that the players adopt a mixed strategy at $t = 1$ in a T-period game, the players will also adopt a mixed strategy in the initial period of any longer game. In terms of the parameter values, this condition can be simplified to $v_L + v_F > (1+\delta)v_S$. Proposition 2 summarizes this discussion.

Proposition 2. *Consider a game with a symmetric mixed-strategy equilibrium with T periods remaining ($x_T < v_L$). Provided $v_L + v_F > (1+\delta)v_S$, the Nash equilibrium at $t = 1$ in any game with a longer horizon is also a mixed-strategy equilibrium.*

Proof. From Figure 3, it is easy to see that condition $z_1 < \delta v_L$ is equivalent to condition $f(\delta v_L) < v_L$, which is

$$\frac{v_L(v_F - \delta v_S)}{v_L + v_F - v_S - \delta v_L}\delta < v_L. \tag{6}$$

Simplification yields the following condition:

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[15] To simplify our presentation, we ignore the equality; both equations yield the same results when $x_T = v_L$.

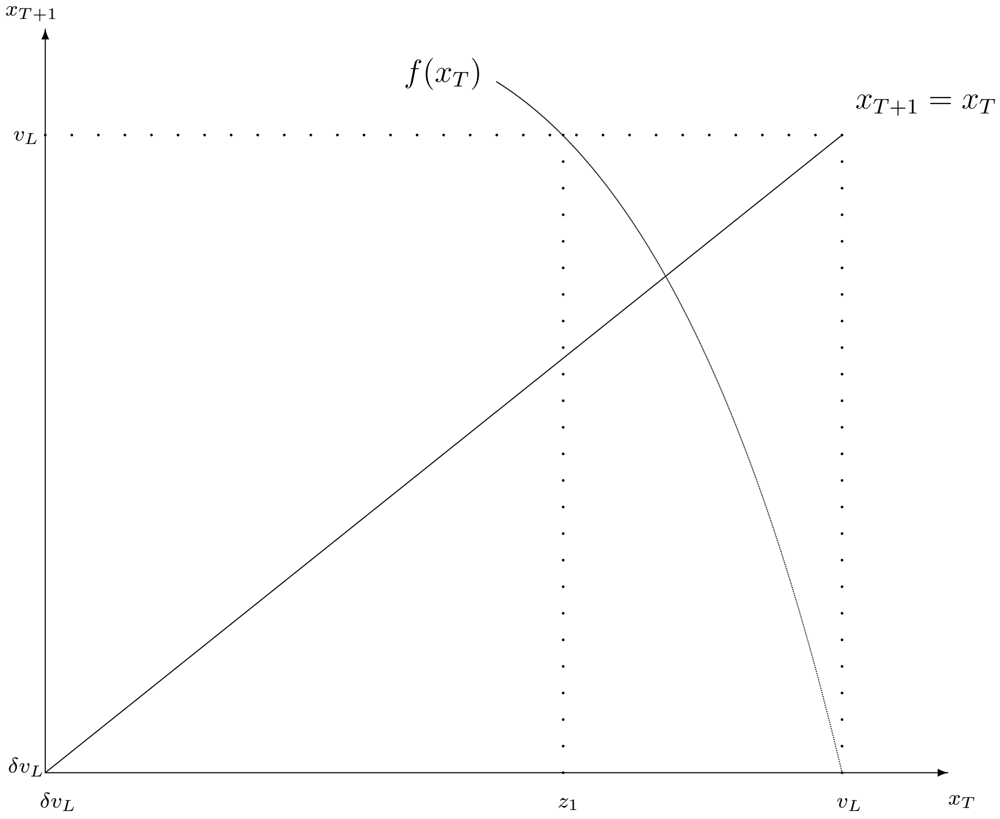

Figure 3: Players' payoff x_{T+1} as a function of x_T

$$v_L + v_F > (1 + \delta)v_S. \ \square \tag{7}$$

When condition 7 is not satisfied, in the zone $\delta v_L < x_T < z_1$, the trajectories will create a value of x_{T+1} higher than v_L; as $x_{T+1} > v_L$ the players have a dominant strategy to wait at $t = 1$ with $T + 1$ periods remaining. In this case, if the players face a game of T periods both will adopt a mixed strategy and invest immediately with some probability in that period (with T periods remaining). However, if there are $T + 1$ periods until the final deadline, in equilibrium at $t = 1$ both players will opt to wait. The reason is that the mixed-strategy payoff with T periods remaining (appropriately discounted) is sufficiently large so that both players have a dominant strategy to wait. Consequently, as the time horizon until the deadline is extended the Nash equilibrium in the initial period ($t = 1$) of the game changes from a prisoners' dilemma equilibrium, to a mixed-strategy equilibrium for one period, then back to a prisoners' dilemma equilibrium. We now consider such a switching game. For ease of exposition, let us construct a function that will correspond to two acts in which there is one mixed-strategy equilibrium followed by one prisoners' dilemma equilibrium.

Consider the case when with T periods remaining each player adopts a mixed strategy. The expected payoff to each party in this period is $\frac{v_L v_F - v_S x_T}{v_L + v_F - v_S - x_T}$. The anticipated payoff in period $T+1$ from waiting is then the T-period mixed-strategy payoff, appropriately discounted. Provided this is greater than v_L each player has a dominant strategy to wait at $t = 1$ with a potential horizon of $T+1$. Now consider when the horizon is one period longer (that is $T+2$ periods). In this case the payoff of waiting at $t = 1$ with $T+2$ periods remaining is equal to the payoff at $t = 1$ with a T deadline, discounted by δ^2 because of the two-period wait. This gives the following function:

$$x_{T+2} = \frac{v_L v_F - v_S x_T}{v_L + v_F - v_S - x_T}\delta^2. \tag{8}$$

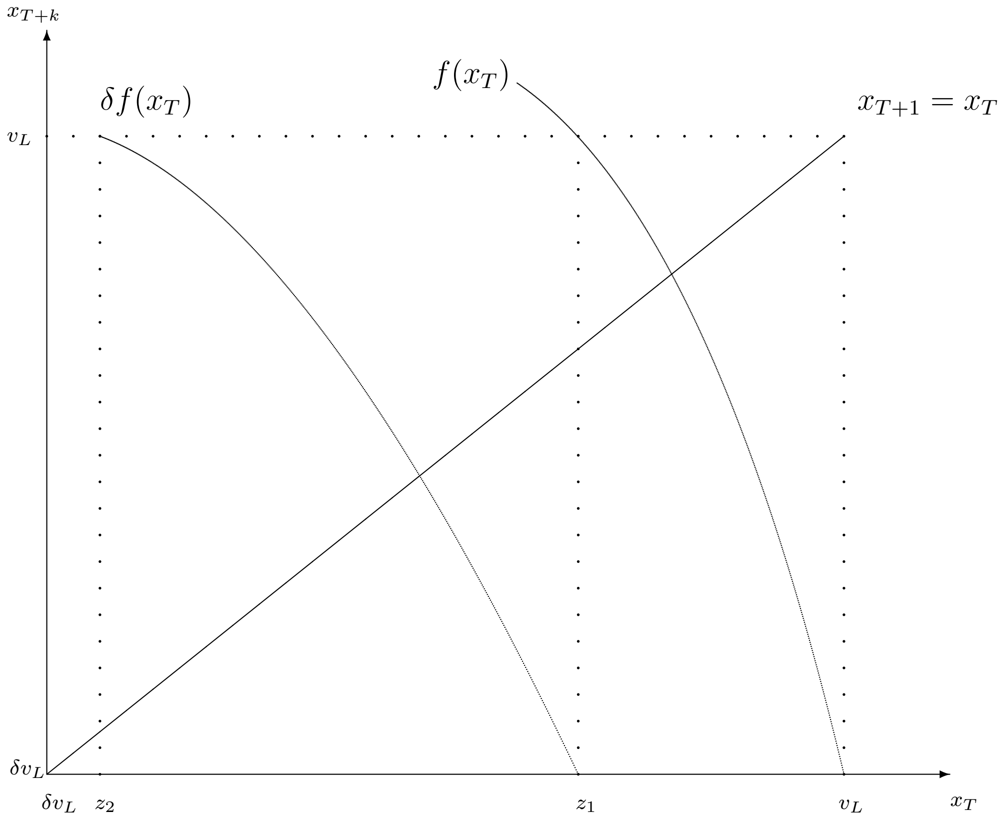

Figure 4: Switching between a dominant strategy to wait and a mixed strategy

The above discussion is illustrated in Figure 4. Define z_2 as $(\frac{v_L v_F - v_S z_2}{v_L + v_F - v_S - z_2})\delta^2 = v_L$. In the zone from z_2 to z_1 function 8 is used. Specifically, with a T-period deadline and when $z_1 > x_T > z_2$ the players will adopt a mixed strategy at $t = 1$ as $x_T < v_L$. If the deadline is $T+1$ periods, however, the players will face a prisoners' dilemma at $t = 1$; as $x_{T+1} > v_L$ both players have a dominant strategy to wait anticipating the pending mixed-strategy payoff with the shorter deadline horizon. If the game instead has a $T+2$ deadline the players will again adopt a mixed strategy in equilibrium, as $x_{T+2} < v_L$. Thus, the players' optimal strategies change with the time until the deadline: in this example, in the zone from z_2 to z_1 game switches from a prisoners' dilemma to a hawk-dove game and back to a prisoners' dilemma as the deadline is extended.

The same methodology applies in the case when $x_T < z_2$. Again, we can construct additional zones in which a mixed-strategy equilibrium is followed by more than one prisoners' dilemma equilibrium. Moreover, the dynamics in k-th zone will be represented by the following function:

$$x_{T+k} = \frac{v_L v_F - v_S x_T}{v_L + v_F - v_S - x_T}\delta^k. \tag{9}$$

When $x_{T+k} = v_L$ and with a sufficiently large k, $x_T < \delta v_L$. At this stage we have divided the $[\delta v_L, v_L]$ region into k zones.

Note that the number of zones - that is $[z_1, v_L]$, $[z_2, z_1]$, and so on - is determined by the parameter values of the particular game. Moreover, there can be any finite number of zones.

Further, in every zone a decreasing function given by equation 21 intersects with an increasing function $x_{T+k} = x_T$ exactly once. This happens because every function $(x_T, ..., x_{T+k})$ from equation 9 in every zone is monotonically decreasing from v_L to δv_L, as in Figure 4. The intersection in the k-th zone corresponds to an equilibrium when exactly the same hawk-dove game is played after $k-1$ prisoners' dilemmas.

Now consider the probability p_T that a player invests in period $t = 1$ with a deadline T. There is a relationship between p_T and x_T, namely $p_T = \frac{v_L - x_T}{v_F - v_S + v_L - x_T} = \frac{x_{T+1}}{\delta(v_S - v_L)} - \frac{v_L}{v_S - v_L}$ if $x_T < v_L$ and $p_T = 0$ otherwise. With two or more zones, p_T could be equal to zero in a given period for any T, while with one zone $p_T > 0$ after some finite T^*, where T^* is the shortest deadline that has a symmetric mixed-strategy equilibrium at $t = 1$.

This result can be demonstrated in the following Figure 5. In the diagram the probability of investment p_T relates to a game in which $v_F = 1.5$, $v_S = 1$, $v_L = 0.5$, $\delta = 0.9$ and $x_1 = 0.6$. The figure shows p_T for $T = 1, ..., 100$. Once the deadline is sufficiently long that the players place some positive probability on investing (when $T = T^*$), they will always place some positive probability on investing with a longer deadline. Moreover, the probability with which the players invest converges to some level $p^* \approx 0.05$. Now consider the probability of investment q_T in a game with $v_F = 1.1$, $v_S = 1$, $v_L = 0.5$, $\delta = 0.9$ and $x_1 = 0.6$. In this second case the probability of investment q_T jumps from positive to zero as the deadline T is extended.

To conclude this section, briefly reconsider the infinite-horizon game. As shown by Fudenberg and Levine (1983), under general conditions the subgame-perfect equilibria with an infinite horizon can be approximated as the limit of a finite-horizon SPE when T is extended. For each value of $x_1 > 0$ there is a unique trajectory with a finite-horizon; each of these trajectories will also

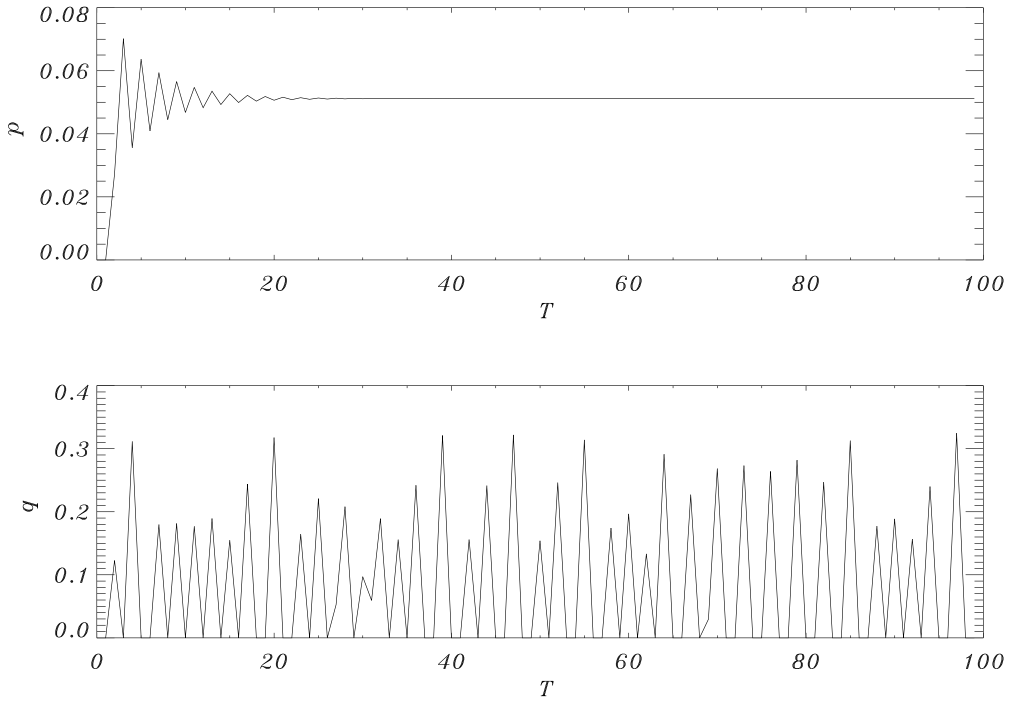

Figure 5: Probabilities of investment p_T and q_T as functions of T

be a subgame perfect equilibrium in the infinite-horizon game. Consequently, there are an infinite number of subgame perfect equilibria in the game with no deadline. However, in this paper we are interested in the unique stationary symmetric equilibrium, which was derived in section 3.

5 Convergence of investment probabilities

In the T-period game, the parties may find themselves in a prisoners' dilemma or in a mixed-strategy equilibrium at $t = 1$ (ignoring non-symmetric equilibria). Further, as demonstrated in Figure 4, the probability of investment can converge as the deadline T is increased or it can show no signs of converging at all. In this section we consider the conditions required for the probability of investment to converge with a finite horizon - as it turns out to the unique stationary symmetric infinite-horizon equilibrium (Proposition 1) - when the number periods until the deadline is increased. More specifically, we address whether there are parameter regions for which there is global convergence - that is, all the trajectories converge to the unique attractor - or if the trajectories move non-periodically.

5.1 Global convergence

If p_T is to converge to some particular probability, both players must be playing a mixed strategy at $t = 1$ with some deadline T and continue to play a mixed strategy with any longer horizon (no switching). This will be the case only if there is one zone. The condition for this is described in Proposition 2:

$$v_L + v_F > (1 + \delta)v_S. \tag{10}$$

The second condition requires that the mixed-strategy equilibrium as T increases converges to some particular value; that is, there are no cycles. This requires the mapping $x_{T+1}(x_T)$ must have a dampened cobweb. For that, the derivative of $f(x_T)$ given by equation 5 must be greater than -1 everywhere. This condition is guaranteed to hold if the derivative at $x_T = v_L$ is greater than -1. After simplification

$$\delta v_L + v_F > (1 + \delta)v_S. \tag{11}$$

It is clear that when the second condition is satisfied, the first is also satisfied. Thus condition 11 is the sufficient condition for the probability of investment in the finite game to converge as the number of periods until the deadline is extended. These conditions are summarized in Proposition 3. An illustration of this discussion is given in Example 1.

Proposition 3. *A sufficient condition for the probability of investment in the finite-horizon game to converge (to the infinite-horizon stationary symmetric equilibrium) is that* $\delta v_L + v_F > (1 + \delta)v_S$.

Example 1. *In this example the probability of investment in the finite-horizon game converges (to the stationary symmetric infinite-horizon equilibrium).*

Let us consider the following example $v_F = 1.5$, $v_S = 1$, $v_L = 0.5$ and $\delta = 0.9$. In this case there is one zone and all the trajectories converge to the unique equilibrium. See Figure 6. □

5.2 Non-convergence

It follows from Result 3 that when the condition $\delta v_L + v_F > (1 + \delta)v_S$ is satisfied the probability of investment is converging when there is an increase in the time until the deadline, and that the outcome of the finite-horizon game converges to the infinite-horizon stationary symmetric equilibrium. This is not always the case, however. The probability of investment as the time horizon is extended can move regularly (for example, move in a cyclical manner as t increases). Alternatively the trajectories can move irregularly (chaotically) as illustrated by the following Example.[16]

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[16] We do not show an example of cyclical behavior to save space, but it is clear that if the system can display chaotic behavior that it can also display any type of regular behavior

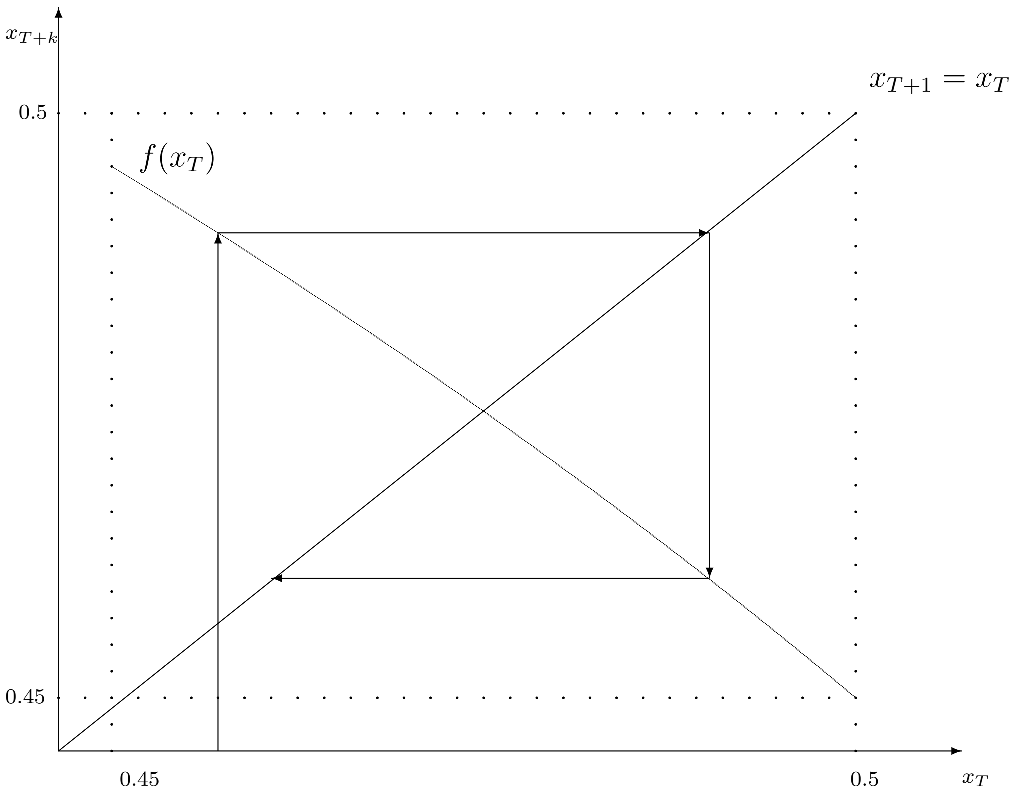

Figure 6: Global convergence

Example 2. *The example illustrates the possibility of chaotic trajectories.*

Let us consider the following example $v_F = 1.1$, $v_S = 1$, $v_L = 0.5$ and $\delta = 0.9$. In this case there are three zones. See Figure 7.

Let us show that all the trajectories are explosive. The largest value of all derivatives for three functions will be derivative of the third function at $x_T = \delta v_L$. This derivative is equal to

$$\frac{\partial x_{T+3}}{\partial x_T} = -\frac{(v_F - v_S)(v_S - v_L)}{(v_L + v_F - v_S - \delta v_L)^2}\delta^3 \approx -1.62. \tag{12}$$

The lowest value of all derivatives for three functions will be derivative of the first function at $x_T = v_L$. This derivative is equal to

$$\frac{\partial x_{T+1}}{\partial x_T} = -\frac{v_S - v_L}{v_F - v_S}\delta = -4.5. \tag{13}$$

Derivatives for values of x_T between δv_L and v_L are monotonically de-

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including cyclical.

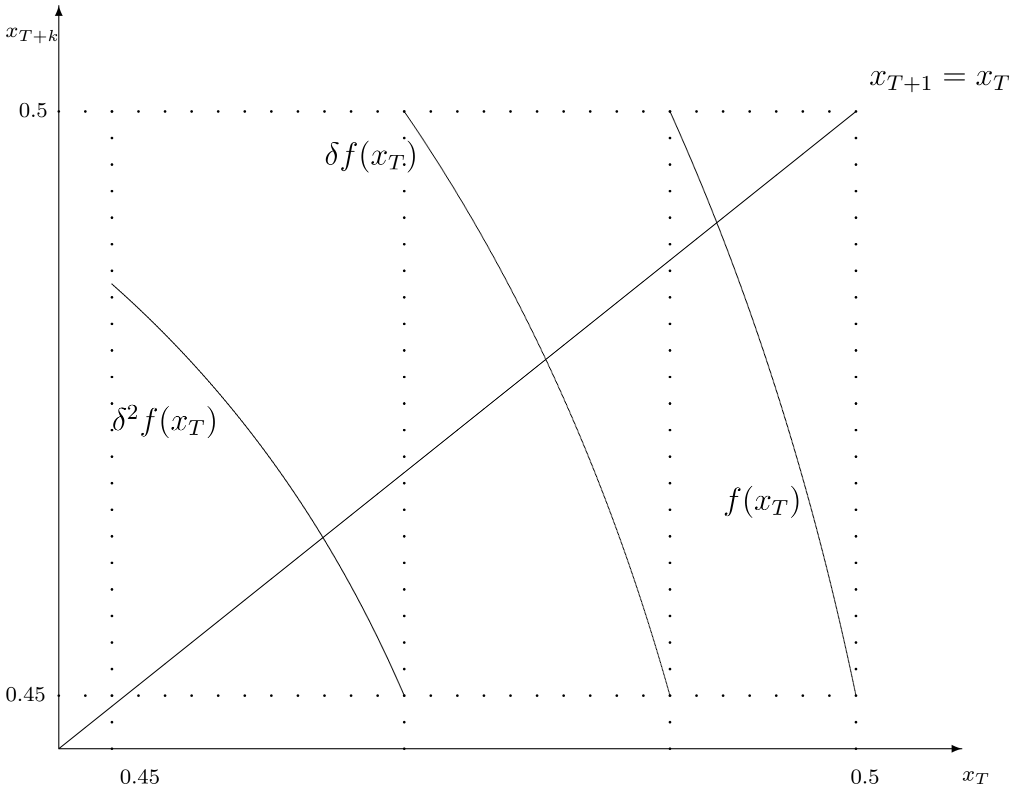

Figure 7: Chaotic behavior

creasing from -1.62 to -4.5. All of them are less than -1, which proves the trajectories are explosive.

To prove that the system with these parameters behaves chaotically we need to show that the system is exponentially unstable or the Lyapunov characteristic exponent (LCE) in this system is positive.[17] By definition, the LCE is

$$\sigma = \lim_{\substack{t \to \infty \\ \Delta(0,x_1) \to 0}} \frac{1}{t} \ln \frac{|\Delta(t,x_1)|}{|\Delta(0,x_1)|}, \tag{14}$$

where $\Delta(0,x_1)$ is a small deviation in the original point, while $\Delta(t,x_1)$ is a resulted deviation in the t-th mapping. The ratio of $\Delta(1,x_1)/\Delta(0,x_1)$ is equal to the value of the derivative of the relevant function at x_1 when $\Delta(0,x_1) \to 0$. Thus, the value of function 14 is just a weighted sum of $\ln(x)$, where $x \in (1.62, 4.5)$. Whatever the weights are, the weighted average always will

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[17] There is no universally accepted definition of chaos. When we have a mapping of a segment into itself, as we have here, the trajectories of a system are chaotic if the system is exponentially unstable. Both non-periodicity and mixing can be shown in this case. See, for example, Elyutin (2000).

be lying in the interval $(\ln(1.62), \ln(4.5))$. Thus, the LCE in this example is positive; consequently the system behaves chaotically. $\square$

In example 2, the probability of investment does not converge as T is extended. Moreover, the probability of investment jumps non-periodically with an increase in the deadline. Following this logic, we can describe sufficient conditions for deterministic chaotic behavior. Namely, chaotic behavior requires that: there is more than one zone (note that this is the opposite to equation 10); and the derivative of $f(x_T)$ given by equation 5 is less than -1 everywhere. The first condition follows from the second because when the slope of $f(x_T)$ is less than -1 everywhere inside the segment $[\delta v_L, v_L]$, this segment will necessarily be divided into more than one zone. Consequently, both of these conditions are satisfied when

$$\delta(v_F - v_S)(v_S - v_L) > (v_L + v_F - v_S - \delta v_L)^2. \tag{15}$$

Proposition 4 reflects the discussion above.

Proposition 4. *A sufficient condition for deterministic chaotic behavior in the finite-horizon game is that* $\delta(v_F - v_S)(v_S - v_L) > (v_L + v_F - v_S - \delta v_L)^2$.

To summarize, in the general case we have divided the space of parameters δ, v_L and v_F into three areas. First, in the area described by condition 11 there is global convergence. Second, in the area described by condition 15 there is chaos. In the third area, in between the other two, there could be regular or irregular behavior. We have not classified this third area.

To help interpret this result, assume that $\delta \to 1$ in the limit. In this case the conditions 11 and 15 are complements - that is the third area in the limit is arbitrarily small. As a consequence, if

$$v_L + v_F < 2v_S \tag{16}$$

there will be chaotic behavior; otherwise, there will be global convergence. In other words, if the average payoff of the leader and follower $(\frac{v_L+v_F}{2})$ is less than the payoff to each player with simultaneous entry (v_S) there will be chaotic behavior. This suggests chaos requires that it is socially optimal for the two parties to enter the market simultaneously, rather than one after the other.

Example 2 shows that the behavior of the players can be sensitive to the number of periods remaining even when T is large. Consider a third party, like a regulator observing two firms considering entering a market or involved in a price-setting game, or an arbitrator observing two bargaining agents. The optimal behavior of the agents depends on the time horizon. Additionally, it also depends on the payoffs in the game with a $T = 1$ deadline. Rather than becoming less important where there is a long potential entry horizon, these $T = 1$ payoffs have an impact on the behavior of the players. Without complete knowledge of the time horizon and the $T = 1$ payoffs, the third party will not be able to predict an agent's action. Further, the probability of an agent making a move will jump in a seemingly irrational manner between periods to an observer who does not possess complete information.

6 An interpretation of the results

It might also he helpful to reconsider this result in light of the motivating example given in section 2. In that context condition 16 requires that

$$\pi_m - K + 2\left[\frac{\delta\pi_d}{1-\delta}\right] < 2\left[\pi_d - \frac{K}{2} + \frac{\delta\pi_d}{1-\delta}\right]$$

or that

$$\pi_m < 2\pi_d. \tag{17}$$

In other words, in the limit as $\delta \to 1$, chaotic behavior requires that the efficiency effect does not hold (that the one-period monopoly profit is less than the total one-period profits for two duopolists). This could hold when there are significant synergies or spillovers between the firms, so that total profits are higher in a duopoly, despite more competition. This suggests chaotic behavior is more likely to occur in industries in which spillovers are important, like in high-tech areas or the computer software industry. These results are illustrated in Figure 8. From the discussion in section 2, $\pi_m < \pi_d + \frac{K}{2}$ and $\pi_d < \frac{K}{2}$, this gives us the possible range of parameters that maintains the assumed order of payoffs ($v_F > v_S > v_L$). The dashed line in the figure indicates the points where $\pi_m = 2\pi_d$. Above the dashed line $\pi_m > 2\pi_d$; in the limit when $\delta \to 1$, there is convergence (Zone 1). On the other hand, below the dashed line, $\pi_m < 2\pi_d$ and there are chaotic entry trajectories (Zone 2).

Expected delay and chaos

Let us examine the dependance between expected delay and value of payoff when firms enter simultaneously, see Figure 9. The figure is constructed for values $v_F = 2$, $v_L = 1$ and $\delta = 0.995$ - note that this implies that $1 < v_S < 2$. Further, x_1 is allowed to vary between 0 and 1, giving a range for the expected delay. For $v_S < 1.5$, approximately, the expected delay does not depend on v_S and x_1. This is consistent with the continuous time model of Fudenberg and Tirole (1991) in that the simultaneous entry payoff (v_S) is not important. Once v_S is larger than about 1.5 there is a range of expected delays (different expected delays for different values of x_1) and, on average, the expected delay is decreasing in v_S. This decrease is particularly strong when v_S gets close to $v_F = 2$.

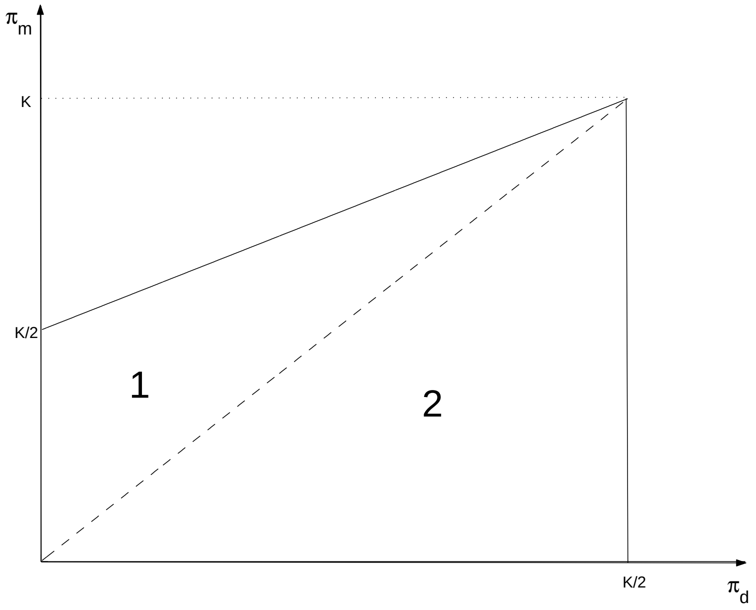

Figure 8: Parameter regions for convergence and chaos

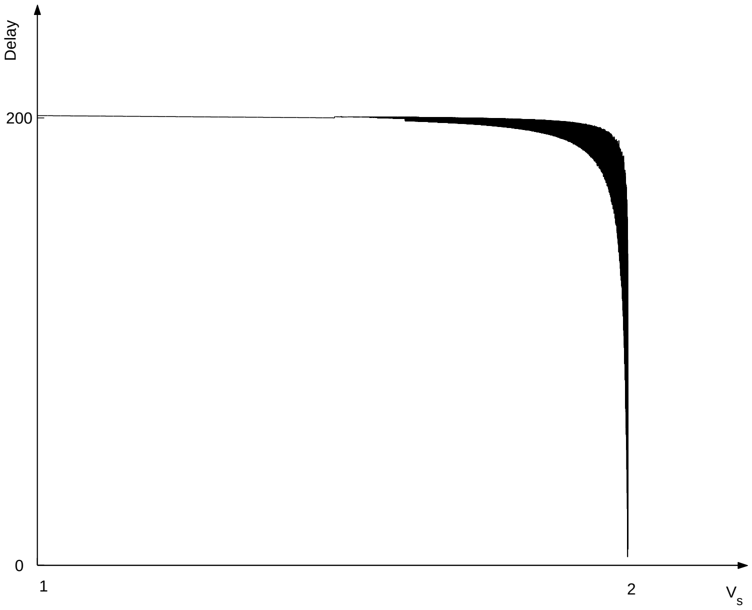

Figure 9: Expected delay and v_s

7 Concluding comments

This paper examines a situation in which there is an advantage from being the second mover. There is essentially a free-rider problem; each party would like the other party to move first, even though immediate investment is required if total surplus is to be maximized. As the deadline, or number of potential waiting periods, is increased the situation can change from one in which both players have a dominant strategy to wait into one in which both parties invest immediately with some positive probability. Perhaps more surprising, with an even longer deadline, it can be the case that the players once again have a dominant strategy to wait. This situation can arise if the mixed-strategy payoff in the game with one less potential investment period is sufficiently high. Consequently, the probability of investment in this market-entry game can be non-monotonic or even chaotic in the number of periods remaining until the deadline. The presence of chaotic trajectories is associated with a smaller expected delay in entry.

Appendix

Derivation of equation 3
In order to derive equation 3 let player B plays strategy I with probability q and strategy W with probability $1-q$. For player A the payoff from playing strategy I is $v_S q + v_L(1-q)$, while the payoff from playing strategy W is $v_F q + x_T(1-q)$. In the mixed-strategy equilibrium these two payoffs are equal, which means

$$q = \frac{v_L - x_T}{v_L + v_F - v_S - x_T}. \tag{18}$$

The mixed-strategy payoff is

$$v_S q + v_L(1-q) = v_L + (v_S - v_L)q = v_L + \frac{(v_S - v_L)(v_L - x_T)}{v_L + v_F - v_S - x_T} = \frac{v_L v_F - v_S x_T}{v_L + v_F - v_S - x_T}. \tag{19}$$

Consequently, the expected payoff from opting to W with a deadline of $T+1$ is the mixed-strategy payoff in the first period of the T-period game, appropriately discounted, or that

$$x_{T+1} = \frac{v_L v_F - v_S x_T}{v_L + v_F - v_S - x_T}\delta \tag{20}$$

as in equation 3.

Demonstration that the number of zones can be any finite integer
To prove this roughly assume that the k-th function passes through point

$[\delta v_L, v_L]$. Consequently, we get the following equation for k:

$$v_L = \frac{v_L v_F - \delta v_L v_S}{v_L + v_F - v_S - \delta v_L} \delta^k. \tag{21}$$

Let us assume that v_F is very close to v_S and substitute $v_F = v_S + \varepsilon$ into equation 21, we get the following equation

$$v_L = \frac{v_S + \varepsilon/(1-\delta)}{1 + \varepsilon/v_L(1-\delta)} \delta^k. \tag{22}$$

For any δ there is a small v_L and a small ε such that k satisfying equation 22 can become arbitrarily large. This shows that the number of zones can take any finite integer.

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